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TO
HYDRO-STATICS

S. L. LONEY

(Elements of Hydro-Statics)

By

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EXAMPLE I.

1. Area of the larger piston $= \pi\ (25)^2 = 625\pi$

Area of the smaller piston $= \pi\ (2)^2 = 4\pi$

From the principle of Hydraulic press

$$\frac{\text{Pressure on larger piston}}{\text{Pressure on smaller piston}} = \frac{\text{Area of the larger piston}}{\text{Area of the smaller piston}}$$

$$\text{Pressure on larger piston} = \frac{625\ \pi}{4\pi} \times 1 \text{ Kilos.}$$

$$= 156{\cdot}25 \text{ Kilos.}$$

Hence the larger piston will support a wt. $= 156{\cdot}25$ Kilos.

2. Area of the larger piston $= 100$ square inches.

Area of the smaller piston $= \frac{1}{4}$ square inches.

$$\frac{\text{Pressure on the smaller piston}}{\text{Pressure on the larger piston}} = \frac{\text{Area of the smaller piston}}{\text{Area of the larger piston}}$$

$$\text{Pressure on the smaller piston} = \frac{1 \times 2240 \times 1}{4 \times 100}$$

$$= 5{\cdot}6 \text{ lbs. wt.}$$

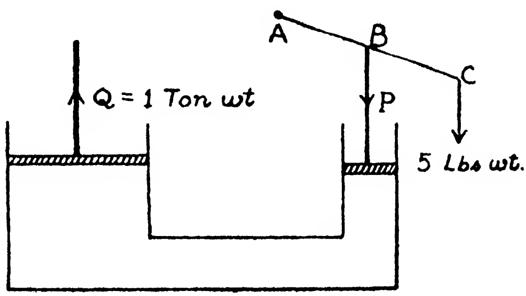
Hence the reqd. force applied $= 5{\cdot}6$ lbs. wt.

3. If W be the weight, then we have

$$W \times 4 \times 144 = 1500$$

$$W = \frac{1500}{576} = 2\frac{29}{48} \text{ lbs.}$$

4. Let P be the pressure applied to the smaller piston, so that



$$\frac{\text{Pressure on the smaller piston}}{\text{Pressure on the larger piston}}=\frac{\text{Area of the smaller piston}}{\text{Area of the larger piston}}$$

$$\text{Pressure on the smaller piston}=\frac{\pi r^2}{\pi(8r)^2}\times 1\times 2240 \text{ lbs. wt.}$$

$$P=35 \text{ lbs. wt.}$$

For the lever AC,

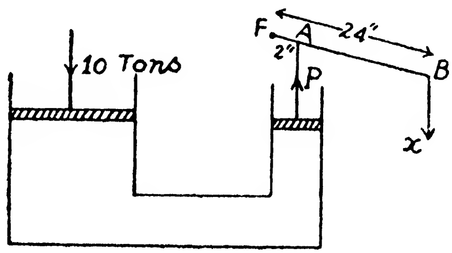
P.AB=5.AC.

35.AB=5.AC

$$\frac{AC}{AB}=\frac{35}{5}=\frac{7}{1}$$

$\therefore$ ratio of arms=7 : 1

5. Let P be the thrust on the smaller piston and x on the end of the lever.



$$\frac{\text{Thrust on the smaller piston}}{\text{Thrust on the larger piston}}=\frac{\text{Area of the smaller piston}}{\text{Area of the larger piston}}$$

$$\therefore \quad P=\frac{\pi(3)^2}{\pi(72)^2}\times 10=\frac{9\times 10}{5184}=\frac{10}{576} \text{ tons wt.}$$

$$x\times FB=P\times FA$$

$$x=\frac{P\times 2}{24}=\frac{10\times 2240}{12\times 576}$$

$$=3\frac{13}{54} \text{ lbs. wt.}$$

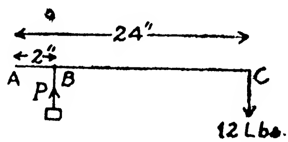
Greatest weight

$$=150\times\pi\,.\,(72)^2=\frac{22}{7}\times\frac{150\times 5184}{2240} \text{ tons wt.}$$

$$=1091\frac{1}{49} \text{ tons wt.}$$

6. The question easily follows from Pascal's law of transmissibility of fluid pressure. Suppose that a pressure p is applied upon the fluid inside the vessel by a slight blow on its cork. By Pascal's law, this pressure is transmitted into the fluid, its amount remaining unaltered. Now the area at the mouth was small but that of the remaining vessel with which the water is in contact is very large. Hence the total force expressed by the vessel is very large. This accounts why a slight blow on the air tight cork of a vessel fully filled with water generally breaks the vessel.

7. Let P be the reqd. pressure, then



$$P \, . \, AB = 12 \, . \, AC$$

$$P \, . \, 2 = 12 \, . \, 24$$

$$P = 144 \text{ lbs. wt. per sq. inch.}$$

8. Area of the circular safety-valve $= \pi \left(\frac{1}{16}\right)^2$ sq. inch.

Let P be the reqd. pressure of the steam, then

$$P \, . \, \pi \left(\frac{1}{256}\right) = \tfrac{1}{2}$$

$$P = \frac{256 \times 7}{2 \times 22} = \frac{64 \times 7}{11}$$

$$= \frac{448}{11} = 40 \frac{8}{11} \text{ lbs. wt. per sq. inch.}$$

9. Let P be the pressure of the steam which is just sufficient to lift the safety-valve.

Hence $P \, . \, \frac{1}{5} = 16.$

$\therefore$ $P = 80$ lbs. wt. per sq. inch.

EXAMPLES II

1. We know that

$$W = V s w$$

$w = 62\frac{1}{2}$ lbs ; $s = 9$; $V = 1$ c. ft.

Therefore,

$$W=1\times 9\times \frac{125}{2}$$

$$=\frac{1125}{2}$$

$$W=562\tfrac{1}{2} \text{ lbs. wt.}$$

2. Specific gravity of brass $=\dfrac{\text{density of brass}}{\text{density of water}}$

$\therefore$ density of brass $=8\times 1000$ ozs. per c. ft.

$$=8000 \text{ ozs. per c. ft.}$$

$$=\frac{8000}{12\times 12\times 12} \text{ ozs. per c. inches.}$$

$$=\frac{8000}{1728}$$

$$=4{\cdot}629\ldots \text{ozs. per c. inches.}$$

3. Sp. gr. of mercury $=\dfrac{\text{wt. of a gallon of mercury}}{\text{wt. of a gallon of water}}$

Wt. of a gallon of mercury $=10\times 13{\cdot}598$

$$=135{\cdot}98 \text{ lbs. wt.}$$

4. Reqd. weight

$$=13{\cdot}6\times 1000 \text{ grammes wt.}$$

$$=13600 \text{ grammes wt.}$$

5. Let the sp . gr. of quartz is s

We know that $W=V\,s\,w$

$\therefore$ $96\tfrac{1}{4}\times s=13\times 19{\cdot}25$

$$s=\frac{13\times 19{\cdot}25\times 4}{385}$$

$$=\frac{13}{5}$$

$$=2{\cdot}6.$$

6. We know that

$$W=V\,.\,s\,w$$

Here W=2240 lbs., $s=19{\cdot}25$, $w=62\frac{1}{2}$ lb.
Therefore,

$$2240 = V \times 19{\cdot}25 \times 62{\cdot}5$$

$$\therefore \quad V = \frac{2240}{19{\cdot}25 \times 62{\cdot}5}$$

$$= 1\frac{237}{275} \text{ c. ft.}$$

7. Suppose V is the volume of a kilogramme of cast copper and V′ is the volume of the same amount of copper wire, therefore,

$$W = V\,s\,w.$$

$$1000 = V \times 8{\cdot}88 \;\ldots\ldots\; \mathrm{I}$$

and $$1000 = V' \times 8{\cdot}79 \;\ldots\ldots\; \mathrm{II}$$

$\therefore$ Reqd. difference of volume

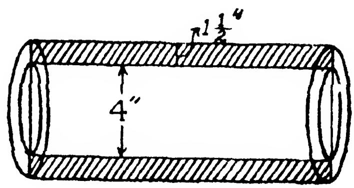
$$= V' - V = \frac{1000}{8{\cdot}79} - \frac{1000}{8{\cdot}88}$$

$$= 1000\left[\frac{1}{8{\cdot}79} - \frac{1}{8{\cdot}88}\right]$$

$$= \frac{1000 \times {\cdot}09}{8{\cdot}79 \times 8{\cdot}88} = \frac{90}{8{\cdot}79 \times 8{\cdot}88}$$

$$= 1{\cdot}153\ldots \text{c. cm.}$$

8. Area of the cross section of the pipe,



$$= \pi\left[\left(\tfrac{7}{2}\right)^2 - (2)^2\right] \text{ sq. in.}$$

$$= \pi\left[\tfrac{49}{4} - 4\right]$$

$$= \frac{33\pi}{4 \times 144} \text{ sq. ft.}$$

$$= \frac{11\,\pi}{192}$$

$$\text{Vol. of the pipe} = \frac{11\,\pi}{192} \times 1 \text{ cubic ft.}$$

$$W = V\,s\,w$$

$$64{\cdot}4 = \frac{11\pi}{192} \times s \times 62\tfrac{1}{2}$$

$$s = \frac{644 \times 192 \times 2 \times 7}{10 \times 11 \times 22 \times 125}$$

$$= 5{\cdot}72 \ldots\ldots\ldots\ldots$$

9. Let the area of cross-section of the rod is A sq. ft.

$\therefore$ Volume of the rod $= \frac{3}{2}$ A cubic feet.

We know that

$$W = V \,.\, s\,w$$

$$3 = \tfrac{3}{2}\,A \times 8{\cdot}8 \times 1000$$

$$A = \frac{3 \times 2}{3 \times 8{\cdot}8 \times 1000}$$

$$= \frac{2}{8800} \text{ sq. ft.}$$

$$= \frac{2 \times 144}{8800}$$

$$= \frac{9}{275} \text{ sq. inch.}$$

10. Volume of the sphere $= \frac{4}{3}\pi r^3$

$$= \tfrac{4}{3}\pi \,.\, (45)^3$$

Let the density of the sphere is ρ. Therefore

$$2376 \times 1000 = \tfrac{4}{3}\pi \times 45 \times 45 \times 45 \times \rho$$

$$\therefore \quad \rho = \frac{2376 \times 1000 \times 3 \times 7}{4 \times 22 \times 45 \times 45 \times 45}$$

$$= \frac{56}{9} = 6\tfrac{2}{9}$$

11. We know that

$$139{\cdot}0625 \times 1000 = V \times 8{\cdot}9$$

Let V is the volume in c. cms.

$$\therefore \quad V = \frac{1390625}{89}$$

$$= 15625 \text{ c. cms.}$$

12. $\text{Density} = \dfrac{\text{Mass}}{\text{Volume}}$

$$= \frac{4900}{9} \text{ lbs. per cubic ft.}$$

$$= \frac{4900}{9}$$

$$= \frac{4900}{9} \times {\cdot}01602 \text{ grammes per c. cm.}$$

$$= 8{\cdot}722 \text{ grammes per c. cm.}$$

13. $\text{Density} = \dfrac{\text{Mass}}{\text{Volume}}$

$\text{Mass} = 36000 \times 1000$ grammes.

$\text{Volume} = 46 \times 10^2 \times 10^2 \times 10^2$ cubic cm.

$$\text{Density} = \frac{36000 \times 1000}{45 \times 100 \times 100 \times 100}$$

$$= \frac{36}{45} \times 62{\cdot}5 \text{ lb. per cubic ft.}$$

$$= 49{\cdot}9 \ldots\ldots \text{ lbs. per cubic ft.}$$

14. Let V is the volume of casting, V′ is the proper volume, therefore,

$$V \times 6{\cdot}3 = V' \times 7{\cdot}5$$

$$\frac{V - V'}{V} = 1 - \frac{V'}{V} = 1 - \frac{63}{75} = \frac{12}{75}$$

Hence the reqd. percentage $= \dfrac{12}{75} \times 100 = 16$.

EXAMPLES III

1. We know that specific gravity of Mixture is given by

$$S = \frac{V_1 s_1 + V_2 s_2}{V_1 + V_2}$$

or

$$\cdot85 = \frac{V_1 \times 1 + V_2 \times {\cdot}8}{V_1 + V_2}$$

or $$\frac{17}{20}=\frac{V_1+\frac{4}{5}V_2}{V_1+V_2}$$

or $$17V_1+17V_2=20V_1+16V_2$$

or $$3V_1=V_2$$

$$\therefore \quad \frac{V_1}{V_2}=\frac{1}{3}.$$

2. We know that

$$\text{Sp. gr. of Mixture } (s)=\frac{W_1+W_2}{\frac{W_1}{s_1}+\frac{W_2}{s_2}}$$

or $$S=\frac{12+20}{\frac{12}{1 \cdot 1}+\frac{20}{\cdot 9}}$$

or $$S=\frac{32\times 1\cdot 1\times \cdot 9}{10\cdot 8+22}$$

$$=\frac{32\times 1\cdot 1\times \cdot 9}{32\cdot 8}$$

Sp. gr. of Mixture $=\cdot 965...$

3. Let ρ be the density of the Mixture

Therefore,

$$\rho=\frac{39\times \cdot 9+51\times \cdot 75}{39+51}$$

$$=\cdot 815.$$

4. Let W_1 oz. of water be added to the solution

We know that

$$S=\frac{W_1+W_2}{\frac{W_1}{s_1}+\frac{W_2}{s_2}}$$

or $$1\cdot 05=\frac{W_1+27}{\frac{W_1}{1}+\frac{27}{1\cdot 08}}$$

or $$\left(W_1+\frac{27}{1\cdot08}\right)(1\cdot05)=W_1+27$$

which, on simplification, gives

$$W_1=15 \text{ ozs.}$$

5. Let W_1 be the mass of wood whose sp. gr. is ·5.

We know that

$$\text{Sp. gr. of Mixture } S=\frac{W_1+W_2}{\frac{W_1}{s_1}+\frac{W_2}{s_2}}$$

$$1=\frac{W_1+500}{\frac{W_1}{\cdot5}+\frac{500}{7}}$$

or $$2W_1+\frac{500}{7}=W_1+500$$

or $$W_1=500-\frac{500}{7}$$

$$=\frac{3000}{7} \text{ ozs.}$$

Let V_1 is the volume of the wood of mass W_1. then

$$W_1=V_1\, s\, w$$

$$\frac{3000}{7}=V_1\times\cdot5\times1000$$

Therefore,

$$V_1=\frac{3\times10}{7\times5}=\frac{6}{7}$$

$$V_1=\frac{6}{7} \text{ c. ft.}$$

6. Let V_1 and V_2 be the volumes of each component of the alloy (i. e. zinc and copper respectively).

$\therefore$ $$V_1+V_2=452 \text{ ...... I}$$

and $$7V_1+8\cdot9V_2=3373 \text{ ... II}$$

To solve I and II, multiply I by 7 and subtract from II

$$1 \cdot 9V_2 = 3373 - 3164$$
$$1 \cdot 9V_2 = 209$$
$$V_2 = \frac{209 \times 10}{19}$$
$$V_2 = 110$$
$$\therefore \quad V_1 = 452 - 11$$
$$= 342.$$

7. Let V be the volume of liquid in each vessels. When B is filled from A the density of the liquid in it is given by

$$= \frac{V\rho_1 + V\rho_2}{V+V} = \frac{\rho_1 + \rho_2}{2}$$

Densities of liquids in B and C are $\frac{\rho_1+\rho_2}{2}$ and ρ_3 respectively and when C is filled from B, the density of the liquid in C is given by

$$= \frac{V \cdot \frac{1}{2}(\rho_1+\rho_2) + V \cdot \rho_3}{V+V} = \frac{\rho_1+\rho_2+2\rho_3}{4}$$

8. We know that

$$S = \frac{V_1 s_1 + V_2 s_2}{V_1 + V_2}$$

Equal volumes of two substances are mixed, $V_1 = V_2$

$$4 = \frac{s_1 + s_2}{2}$$

$$\therefore \quad s_1 + s_2 = 8 \text{ ...... I}$$

Also, $$S = \frac{W_1 + W_2}{\frac{W_1}{s_1} + \frac{W_2}{s_2}}$$

Equal weights are mixed, $W_1 = W_2$

$$3 = \frac{2s_1 s_2}{s_1 + s_2} = \frac{2s_1 s_2}{8}$$

$$s_1 s_2 = 12 \text{ ...... II}$$

Solving I and II, we get

$$s_1=6, s_2=2.$$

9. We know that

$$S=\frac{V_1 s_1+V_2 s_2}{n(V_1+V_2)}$$

$$n=\frac{96}{100}$$

Equal volumes of alcohol and distilled water are mixed. Therefore $V_1=V_2=V_1$

$$\therefore \qquad S=\frac{s_1+s_2}{2n}=\frac{1+\cdot 8}{2\times\frac{96}{100}}$$

$$=\frac{1\cdot 8\times 100}{2\times 96}=\cdot 9375$$

Sp. gr. of Mixture $=\cdot 9375$.

10. We know that

$$S=\frac{V_1 s_1+V_2 s_2}{n(V_1+V_2)}$$

$$1\cdot 615=\frac{7\times 1\cdot 843+3\times 1}{n(7+3)}$$

$$10\times 1\cdot 615\, n=12\cdot 901+3$$

$$n=\frac{15\cdot 901}{16\cdot 15}$$

$\therefore$ Amount of contraction $=10\times(1-n)$

$$=10\left(1-\frac{15\cdot 901}{16\cdot 15}\right)$$

$$=\frac{10\times\cdot 249}{16\cdot 15}$$

$$=\frac{249}{1615}\text{ c. cms.}$$

11. The sp. gr. of A and B be s_1 and s_2 respectively. Let the amount of A be m lbs. Using the usual formula

$$s=\frac{m+n}{\dfrac{m}{s_1}+\dfrac{n}{s_2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{ I}$$

$$s'=\frac{m+2n}{\dfrac{m}{s_1}+\dfrac{2n}{s_2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{ II}$$

$$s''=\frac{m+3n}{\dfrac{m}{s_1}+\dfrac{3n}{s_2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{ III}$$

From I,

$$s\left(\frac{m}{s_1}+\frac{n}{s_2}\right)=m+n$$

or $$m\left(\frac{s}{s_1}-1\right)=n\left(1-\frac{s}{s_2}\right) \quad \ldots\ldots\ldots\ldots\ldots \text{ IV}$$

From II,

$$s'\left(\frac{m}{s_1}+\frac{2n}{s_2}\right)=m+2n$$

$$m\left(\frac{s'}{s_1}-1\right)=2n\left(1-\frac{s'}{s_2}\right) \quad \ldots\ldots\ldots\ldots \text{ V}$$

From III,

$$s''\left(\frac{m}{s_1}+\frac{3n}{s_2}\right)=m+3n$$

$$m\left(\frac{s''}{s_1}-1\right)=3n\left(1-\frac{s''}{s_2}\right) \quad \ldots\ldots\ldots\ldots \text{VI}$$

Dividing IV by V,

$$\frac{s-s_1}{s'-s_1}=\frac{s_2-s}{2(s_2-s')} \text{ or } 2(s-s_1)(s_2-s')=(s_2-s)(s'-s_1)$$

Dividing IV by VI

$$\frac{s-s_1}{s''-s_1}=\frac{s_2-s}{3(s_2-s'')}$$

or $3(s-s_1)(s_2-s'')=(s_2-s)(s''-s_1)$

EXAMPLES IV

1. If w be the weight of a cubic foot of water, then the required pressure is given by

$$p=w\,.\,h$$
$$=\frac{1000}{12\times 12\times 12}\times 1760\times 3\times 12$$
$$=\frac{1000\times 110}{3\times 16}\text{ lbs. wt.}$$
$$=\frac{6875}{3}$$
$$=2291\tfrac{2}{3}\text{ lbs. wt.}$$

2. Let x be the reqd. depth of water in inches, where the pressure is 100 lbs. wt. per sq. inch. Then from the usual formula,

$$p=\pi+\text{wh}.$$
$$100=15+\frac{1000\,.\,x}{12\times 12\times 12\times 16}$$
$$\therefore\quad \frac{1000}{1728\times 16}x=100-15=85$$
$$x=\frac{85\times 1728\times 16}{1000}\text{ inches.}$$

reqd. depth$=195{\cdot}84$ ft.

3. Let the depth of the reqd. point be x feet, then

$$p=w\,.\,x$$
$$12090=1{\cdot}56\times 1000\times x$$
$$x=\frac{12090}{1560}=\frac{31}{4}$$
$$\therefore\quad x=7\tfrac{3}{4}\text{ ft.}$$

4. Let the height of the third floor above the basement is x inches.

Pressure at the basement=Pressure at the third floor$+w\,.\,x$

$$34=18+\frac{125}{2}\cdot\frac{1\,.\,x}{12\times 12\times 12}$$

$$\frac{125}{2\times 1728}\,x=34-18=16.$$

$$x=\frac{16\times 2\times 1728}{125}\ \text{inches}$$

$$x=\frac{16\times 2\times 1728}{125\times 12}\ \text{ft.}$$

$$=36{\cdot}864\ \text{ft.}$$

5. Let x be the required height of a column of air, hence

$$p=w\ .\ x$$

$$14=x\times {\cdot}00125\times\frac{125}{2\times 12\times 12\times 12}$$

Since the weight of 1 cubic foot of water $=\frac{125}{2}$ lbs.

$$\therefore \qquad 14=\frac{x\times 125\times 125}{800\times 2\times 1728}$$

$$x=\frac{14\times 800\times 2\times 1728}{125}\ \text{inches}$$

$$=\frac{14\times 800\times 2\times 144}{125\times 12}\ \text{ft.}$$

$$=4\ \text{miles}\ 1561{\cdot}6\ \text{yds.}$$

6. Required force

$$=\frac{16}{12}\times w\times 34=\frac{4}{3}\times\frac{125}{2}\times 34.$$

$$=2833\tfrac{1}{3}\ \text{lbs. wt.}$$

7. Let x be the depth of well and w the intrinsic weight of water, then

Pressure at the bottom of the well $=(30+x)w$

Pressure at a depth of 2 feet $=(30+2)w=32w$

Hence from the question

$$(30+x)w=4\ .\ 32w$$

$$30+x=128$$

$$x = 128 - 30$$
$$= 98 \text{ feet.}$$

8. Let the depth of the required point be x feet and w the weight of water per unit volume. Then

$$\text{Pressure at the required point} = (34 + x)w$$
$$\text{Pressure at a depth 10 feet} = (10 + 34)w$$
$$= 44w$$

Hence

$$(34 + x)w = 2 \times 44w$$
or
$$34 + x = 88$$
$$\therefore \quad x = 54 \text{ feet.}$$

9. Suppose the atmospheric pressure be that which would be due to a height of x feet of water.

$$\text{Pressure at a depth of 5 feet} = (x+5)w$$
and the pressure at a depth of 44 feet $= (x+44)w$

Therefore from the problem,

$$(x+5)w = \tfrac{1}{2}(x+44)w$$
$$2x + 10 = x + 44$$
$$x = 34 \text{ feet.}$$

$$\therefore \quad \text{Atmospheric pressure} = 34 \times \frac{125}{2} \text{ lbs. wt. per sq. feet.}$$

$$= 17 \times 125 \times \frac{1}{144} \text{ lbs. wt. per sq. inch}$$

$$= 14\frac{109}{144} \text{ lbs. wt. per sq. inch.}$$

10. 1 fathom $= 2$ yards.

Pressure per square yard at a depth of 20 yards

$$= 20 \times 1.026 \times \frac{125}{2} \times 3 \times 3 \times 3 \text{ lbs. wt.}$$

$$= 20 \times 1.026 \times 62.5 \times \frac{27}{2240} \text{ tons wt.}$$

$$= 15.45 \ldots \text{ tons wt.}$$

11. Let the reqd. depth be x metres and w be the wt. of water per unit volume

Hence

$$x \times 13{\cdot}596 \times w = 500w$$

$$x = \frac{500}{13{\cdot}596}$$

$$= 36{\cdot}77 \ldots \text{ metres.}$$

12. Let the required depth be x cms., then from the question

$$13{\cdot}596 \times x \times w = 1000 \times w$$

$$x = \frac{1000}{13{\cdot}596}$$

$$= 73{\cdot}55 \ldots \text{ cm.}$$

13. Reqd. force on the valve = thrust on that area.

Pressure per square cm. = wt. of 750 m.m. of mercury

$$= 75 \times 13{\cdot}6 \text{ grammes wt.}$$

Area of the square valve whose side is one decimeter

$$= 10 \times 10 = 100 \text{ sq. cms.}$$

Therefore,

Reqd. force $= 75 \times 13{\cdot}6 \times 100$

$= 102000$

14. (i) Pressure per square inch at a depth of 10 feet

$$= 15 + 5 \times \frac{125}{12 \times 12}$$

$$= 15 + \frac{625}{144}$$

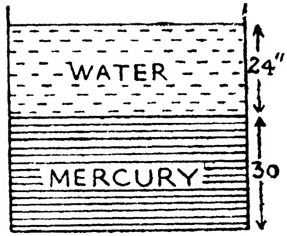
$$= 19{\cdot}34 \text{ lbs. wt.}$$

(ii) Pressure per square inch at a depth of 1 mile

$$= 15 + 1760 \times 3 \times \frac{125}{2 \times 12 \times 12}$$

$$= 2306{\cdot}6 \text{ lbs. wt.}$$

15. Required pressure at the bottom of the vessel in lbs. wt. per sq. inch.

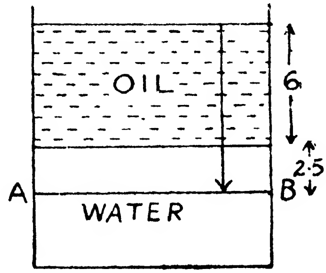


$$=[30\times 13{\cdot}6+24\times 1]\times\frac{125}{2\times 12\times 12\times 12}$$

$$=(408+24)\;\frac{125}{2\times 12\times 12\times 12}$$

$$=\frac{432\times 125}{2\times 12\times 12\times 12}=\frac{125}{8}=15\tfrac{5}{8}\text{ lbs. wt. per sq. inch.}$$

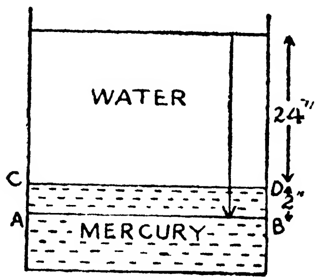
16. Required pressure at the depth 8·5 inches below the upper surface of oil in per sq. inch.



$$=(2{\cdot}5+6\times{\cdot}92)\times 252\text{ grains}$$

$$=2021{\cdot}04\text{ grains wt.}$$

17. Required pressure at a depth of two inches below the common surface in lbs. wt. per sq. inch.



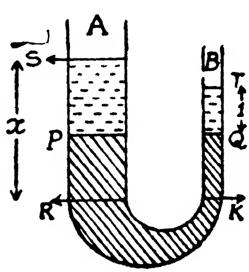
$$=(24+2\times 13{\cdot}568)\times\frac{1000}{12\times 12\times 12}\times\frac{1}{16}$$

$$=1\,\frac{367}{432}\text{ lbs. wt. per sq. inch.}$$

18. Cross section of A tube = 1 sq. inch.

Cross section of B tube = ·1 sq. inch.

Let the levels of mercury in the two limbs be P and Q initially so that P and Q lie in the same horizontal plane.



When the water has been poured in left limb upto S, let the level of mercury rise from Q to T by 1" and

that in left, fall from P to R. Then,

$$PR \times 1 = 1 \times \cdot 1$$

$$\therefore \quad PR = \cdot 1 = KQ.$$

If now, K be a point in the right limb in the same horizontal level as R, then,

$$\Pi + x \,.\, g = \Pi + 1\cdot 1 \times 13\cdot 596 \times g$$

$$x = 1\cdot 1 \times 13\cdot 596$$

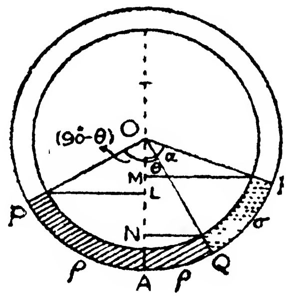
Since the pressure at R and K, being in the same horizontal plane, are equal.

$$\therefore \quad x = 14\cdot 9556 \text{ inch.}$$

Hence the amount of water poured in

$$= 14\cdot 9556 \text{ inch.}$$

20. Let the tube PAQ containing liquid of density ρ such that,



$$\angle PAQ = 90$$

and the tube QR containing liquid of density σ, then that $\angle$ QOR $= \alpha$.

Let the common surface OQ makes an angle θ with the vertical.

Pressure at A due to liquid from left

$$= \rho \,.\, g \,.\, LA = \rho g \; (OA - OL)$$

$$= \rho g \; (a - a \sin \theta)$$

Pressure at A due to liquids from right

$$= \rho g \,.\, NA + \sigma g \,.\, MN$$

$$= \rho g \; (OA - ON) + \sigma g \; (ON - OM)$$

$$= \rho g \; (a - a \cos \theta) + \sigma g \; \{a \cos \theta - a \cos (\alpha + \theta)\}$$

Since the pressure at A due to liquids from left and righ must be the same. Therefore,

$$\rho g a \; (1 - \sin \theta) = \rho a g \,.\, (1 - \cos \theta) + \sigma g a \,.\, [\cos \theta - \cos (\alpha + \theta)]$$

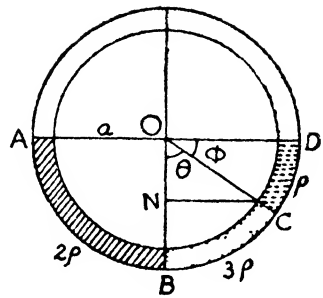
or $\quad \rho - \rho \sin \theta = \rho - \rho \cos \theta + \sigma \cos \theta - \sigma \cos \alpha \,.\, \cos \theta + \sigma \sin \alpha \,.\, \sin$

or $\quad \cos \theta \; (\rho - \sigma + \sigma \cos \alpha) = \sin \theta \; (\rho + \sigma \sin \alpha)$

$$\therefore \qquad \tan\theta = \frac{\rho - \sigma + \sigma\cos\alpha}{\rho + \sigma\sin\alpha}$$

or
$$\theta = \tan^{-1}\frac{\rho - \sigma + \sigma\cos\alpha}{\rho + \sigma\sin\alpha}.$$

21. Let O be the centre of the uniform circular tube and a its radius. Let us suppose that the liquid of density 3ρ occupies the lower position in the right quadrant and that of density ρ occupies the upper position.



Let $\angle$ BOC $= \theta$

and $\angle$ COD $= \varphi$

It is required to prove for us that $\theta = 2\varphi$, because then the volume of the liquid of density 3ρ would be clearly double the volume of the liquid of density ρ.

Considering the equilibrium of B, we have

$$2\rho \,.\, a = 3\rho \,.\, (a - a\cos\theta) + \rho \,.\, a\sin\varphi$$

$$2 = 3 - 3\cos\theta + \sin\varphi$$

$$3\cos\theta - 1 = \sin\varphi$$

From the figure, $\varphi = \frac{\varphi}{2} - \theta.$

$$\therefore \qquad 3\cos\theta - 1 = \sin\left(\frac{A}{2} - \theta\right) = \cos\theta.$$

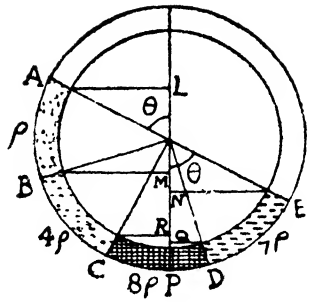
$$\therefore \qquad 2\cos\theta = 1$$

$$\cos\theta = \tfrac{1}{2}$$

$$\theta = 60$$

$$\therefore \qquad \varphi = 30.$$

(Hence the rusult).



22. Let O be the centre and a its radius of the uniform circular tube. Let AB, BC, CD and DE containing liquids of density ρ, 4ρ, 8ρ and 7ρ.

Since the volume of each liquid is equal.

Hence

$$\angle AOB = \angle BOC = \angle COD$$
$$= \angle DOE = 45°.$$

Let the diameter AE inclined at an angle θ with the vertical.

Pressure at P due to liquids from left.

$$= 8\rho \,.\, [a - a\sin\theta] + 4\rho \,.\, \left[a\sin\theta - a\cos\left(\frac{\pi}{4} + \frac{\pi}{2} - \theta\right)\right]$$
$$+ \rho \,.\, \left[a\cos\left(\frac{\pi}{4} + \frac{\pi}{2} - \theta\right) + a\cos\theta\right]$$

Pressure at P due to liquids from right

$$8\rho \,.\, \left[a - a\cos\left(\theta - \frac{\pi}{4}\right)\right] + 7\rho \,.\, \left[a\cos\left(\theta - \frac{\pi}{4}\right) - a\cos\theta\right]$$

Since the pressure at P due to liquids from left and right are equal. Hence

$$8\left[1 - \cos\left(\theta - \frac{\pi}{4}\right)\right] + 7\left\{\cos\left(\theta - \frac{\pi}{4}\right) - \cos\theta\right\}$$
$$= 8(1 - \sin\theta) + 4\left\{\sin\theta - \sin\left(\theta - \frac{\pi}{4}\right)\right\}$$
$$+ \left\{\sin\left(\theta - \frac{\pi}{4}\right) + \cos\theta\right\}$$

or $$8 - \cos\left(\theta - \frac{\pi}{4}\right) - 7\cos\theta = 8 - 4\sin\theta$$
$$- 3\sin\left(\theta - \frac{\pi}{4}\right) + \cos\theta.$$

or $4 \sin \theta + 3 \sin \left(\theta - \frac{\pi}{4}\right) - \cos \left(\theta - \frac{\pi}{4}\right) = 8 \cos \theta.$

or $4 \sin \theta + \frac{3}{\sqrt{2}} (\sin \theta - \cos \theta) - \frac{1}{\sqrt{2}} \cdot (\cos \theta + \sin \theta) = 8 \cos \theta$

or $4 \sin \theta + \sqrt{2} \sin \theta = 2\sqrt{2} \cos \theta + 8 \cos \theta$

or $\sin \theta (4 + \sqrt{2}) = 2 (\sqrt{2} + 4) \cos \theta$

$\therefore \quad \tan \theta = 2$

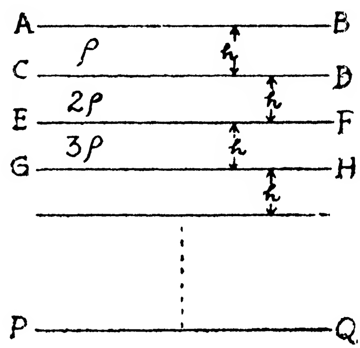
$\therefore \quad \theta = \tan^{-1} 2$ Proved.

23. Pressure at the lowest poin:

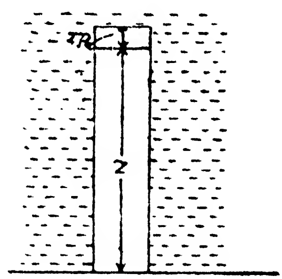
$$= h\rho g + h \cdot 2\rho \cdot g + h \cdot 3\rho g + \ldots + \ldots\ldots h \cdot n\rho g$$

$$= \rho h g \, (1 + 2 + 3 + \ldots + n)$$

$$= \frac{n\,(n+1)}{2} h\rho g.$$



24. Consider a small cylinder of height δz at a depth z below the free surface.



Weight of this cylinder $= \frac{\rho z}{a} \cdot \delta z \cdot g$

Pressure at the point at a depth z

= the atmospheric pressure

+ weight of the cylinder from free surface to a depth z.

$$= \pi + \int_0^z \frac{\rho z}{a} \cdot \delta z g$$

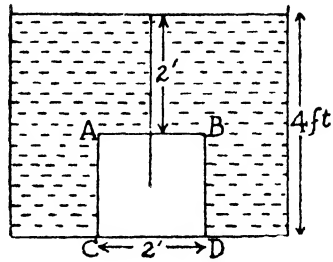
$$= \pi + \frac{\rho g}{a} \left[\frac{1}{2} z^2 \right]_0^z$$

$$= \pi + \frac{1}{2a} \rho g \cdot z^2.$$

EXAMPLES V

1. We know that

thrust = area × pressure at the centre of gravity.



The depth of centre of gravity

$=2+1=3$ ft.

Hence the thrust

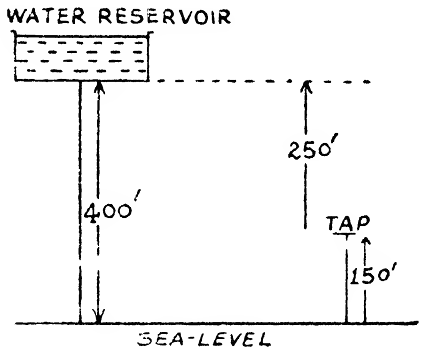
$$=2^2\times3\times\frac{125}{2}\text{ lbs. wt.}$$

$$=6\times125$$

$$=750\text{ lbs. wt.}$$

2. Area of the tap

$$=\frac{3}{2\times144}\text{ sq. ft.}$$



The depth of C. G. of the tap

$$=400-150$$

$$=250.$$

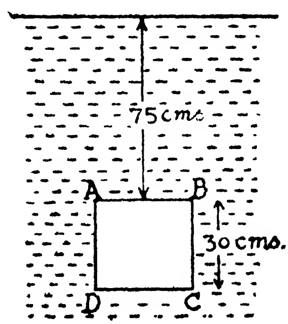
Hence the required thrust on the tap

$$=\frac{3}{2\times144}\times250\times\frac{125}{2}\text{ lbs. wt.}$$

$$=162\frac{73}{96}\text{ lbs. wt.}$$

3. Area of the square

$$=30\times30=900\text{ sq. cm.}$$



Depth of the centre of gravity of vertical face $=75+15$

$$=90\text{ cms.}$$

Hence the thrust on the vertical face $=900\times90$

$$=81000\text{ grammes wt.}$$

The depth of C. G. of the upper face AB=75 cms.

Thrust on that face$=900\times 75$

$=67500$ grammes wt.

The depth of C G. of the lower face CD$=75+30=105$ cms.

Therefore, thrust on this face

$=900\times 105$

$=94500$ grammes wt.

4, Area of the whole hole$=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$ sq. feet. The depth of the whole from the water line=20 ft. Hence the thrust on that whole=force required

Hence the required force

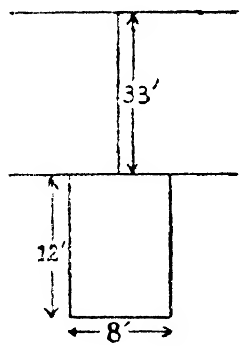
$=\frac{1}{4}\times 20\times 64=5\times 64$

$=320$ lbs. wt.

5. Area of the wall

$=12\times 8$

$=96$ sq. feet.



The depth of the C. G. from the effective surface

$=33+6$

$=39$ feet.

Hence the required thrust

$=96\times 39\times\frac{125}{2}$ lbs. wt.

$=\frac{3\times 39\times 25}{28}$ Tons wt.

$=\frac{3\times 39\times 25}{28}$

$=104\frac{13}{28}$ Tons wt.

6. Area of the base of the vessel$=15\times 15$

$=225$ sq. cm.

The depth of the base of the vessel $=15+7{\cdot}5$

$=22{\cdot}5$ cm.

Hence the thrust on the base of the vessel

$=225\times 22{\cdot}5.$

$=5062{\cdot}5$ grammes wt.

7 Area of the circular disc $=\pi\,.\,(7)^2=49\,\pi$

The depth of the circular disc from the effective surface

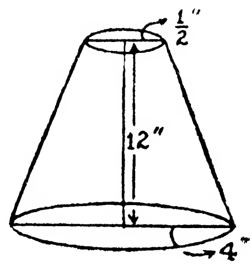
$=1033+5000=6033$ cm.

Hence the thrust on the circular disc

$=49\,\pi\times 6033$

$=295617\,\pi$ grammes wt.

9. Pressure at the centre of the base



$$=\frac{125}{2\times 12\times 12}$$

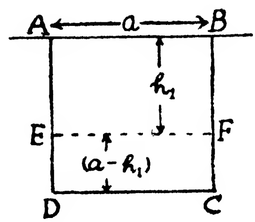
$$=\frac{125}{288}\text{ lbs. wt. per sq. inch.}$$

Hence the required thrust on the base $=\pi\,.\,4^2\,.\,\dfrac{125}{288}$

$$=\frac{125\,\pi}{18}=\frac{125}{18}\times\frac{22}{7}$$

$$=21\,\frac{52}{63}\text{ lbs. wt.}$$

10. Let the side of the square is a and h_1 is the depth of the horizontal line.



Pressure on AEFB

$$=a\times h_1\times\frac{h_1}{2}\times w$$

$$=\frac{a\,h_1^2}{2}\,w.$$

Pressure on EFCD

$$=a\times(a-h_1)\times\left\{h_1+\frac{a-h_1}{2}\right\}\times w$$

$$=a\,(a-h_1)\left(\frac{a+h_1}{2}\right)w$$

$$=\frac{a\,(a^2-h_1^2)\,w}{2}$$

According to the given condition

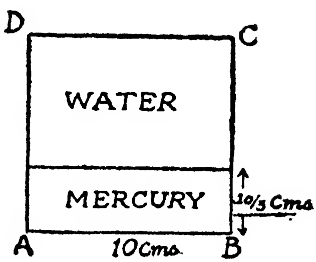
$$\frac{a\,(a^2-h_1^2)\,w}{2}=\frac{a\,.\,h_1^2}{2}\,w$$

$$a^2-h_1^2=h_1^2$$

$$2\,h_1^2=a^2$$

$$\therefore \qquad h_1=\frac{a}{\sqrt{2}}$$

Therefore the line should be drawn parallel to the base at a depth of $\frac{1}{\sqrt{2}}$ times the side of the square from the free surface.



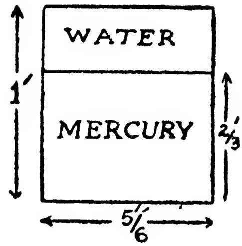
11. ABCD is a square. The thrust of the liquids in the same as if the thrust of water in contact with the whole vessel and the Mercury of sp. gr. (13·6—1) in contact with the lower part only. Hence the reqd. thrust

$$=10.10.5+10\,.\,\tfrac{10}{3}\,\tfrac{10}{6}\,.\,12{\cdot}6$$

$$=100\,(5+7)$$

$$=1200 \text{ grammes wt.}$$

$$=1{\cdot}2 \text{ kilogram}$$



12. The thrust of the liquids is the same as if the thrust of water in contact with the whole vessel and the Mercury of sp. gr. (13·596−1) in contact with the lower part only.

The thrust of water on the whole vessel

$$=\frac{5}{6}\cdot 1\times\tfrac{1}{2}\,w=\frac{5}{12}\,w.$$

The thrust of Mercury on the lower part

$$=\frac{5}{6}\cdot\frac{2}{3}\cdot\frac{2}{6}\cdot(12{\cdot}596)\,w=\frac{5}{27}(12{\cdot}596)\,w$$

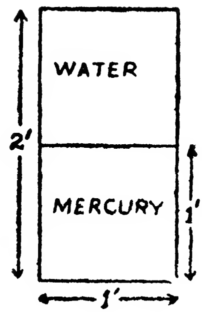
Atmospheric pressure$=10.12.15$ lbs. wt.

Hence the whole pressure

$$=\frac{5}{12}\,w+\frac{5}{27}(12{\cdot}596)\,w+1800 \text{ lbs. wt.}$$

$$=\frac{5\times125}{12\times2}+\frac{5}{27}(12{\cdot}596)\times\frac{125}{2}+1800$$

$=1971{\cdot}8$......lbs. wt.



13. The thrust on the rectangular vessel may be considered due to the water in contact with the whole vessel and the Mercury of sp. gr. (13·5−1) in contact with the lower part only.

Hence the reqd. thrust

$=2.1\,.\,1.w+1.1\,.\,\tfrac{1}{2}\,.\,125\,.\,w$

$$=\frac{33}{4}\times\frac{125}{2}$$

$$=515\,\frac{5}{8}\text{ lbs. wt.}$$

14. Let the depth of the two small areas be x and y feet below the surface of water.

It is given that

$$x \cdot w = 4y \cdot w$$

$$\therefore \quad x = 4y \ldots\ldots \mathrm{I}$$

and $$(x-1)\, w = 9\,(y-1)\, w$$

$$x - 1 = 9\,(y-1) \ldots\ldots \mathrm{II}$$

Solving I and II we get

$$x = \frac{32}{5}\,;\; y = \frac{8}{5}\,.$$

15. Upward thrust on the lid

$$= 1 \times 1 \times 20\, w$$

$$= 20 \times \frac{125}{2}$$

$$= 1250 \text{ lbs. wt.}$$

Downward thrust on the base $= 1 \times 1 \times (20+1) \times \frac{125}{2}$.

$$= 1312\tfrac{1}{2} \text{ lbs. wt.}$$

Reqd. difference $= 1312\frac{1}{2} - 1250$

$$= \frac{125}{2} = \text{wt. of the water in base.}$$

The thrust on the base is to be greater than the wt. of water because it has to balance the wt. of the downward thrust of the lid on water as well as the weight of water.

16. Area of the embankment $= (100 \times 3)\ (88)$ sq. ft.

The depth of C. G. $= 44$ ft.

Total thrust on the embankment

$$= 300 \times 88 \times 44\, w$$

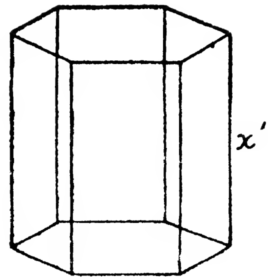
$$= \frac{300 \times 88 \times 44}{3 \times 3 \times 3} \times \tfrac{3}{4} \text{ Tons wt.}$$

$$= 32266\tfrac{2}{3} \text{ Tons weight.}$$

Weight of the water in lake

$$= \tfrac{1}{4} \times 1760 \times 100 \times \tfrac{44}{3} \times \tfrac{3}{4} \text{ Tons.}$$

$$= 484000 \text{ Tons.}$$



17. If x' be the depth of the cistern. The depth of the C. G. of the cistern, below the free surface$=\frac{x'}{2}$. The thrust on the side of the cistern

$$=w \cdot \frac{x}{2} \cdot (x \cdot \sqrt{3}).$$

$$=\frac{\sqrt{3}}{2} x^2 \cdot w$$

Area of the base

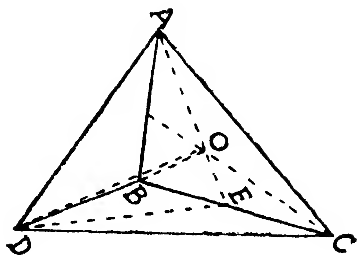
$$=6 \cdot \tfrac{1}{2} \cdot \sqrt{3} \cdot \sqrt{3} \cdot \frac{\sqrt{3}}{2}=\frac{9\sqrt{3}}{2} \text{ sq. ft.}$$

$\therefore$ Thrust on the base$=\frac{9\sqrt{3}}{2} \cdot x \cdot w.$

According to the problem

$$\frac{\sqrt{3}}{2} x^2 w=\frac{9\sqrt{3}}{2} \cdot x \cdot w$$

$$x=9 \text{ feet.}$$



18. Let ABC be the horizontal face of the tetrahedron, and from the opposite vertex D, draw a perpendicular DO upon ABC. Then O is the centre of the base.

$$AO=\tfrac{2}{3} \cdot AE=\tfrac{2}{3} \cdot \frac{\sqrt{3}}{2} a$$

$$=\frac{a}{\sqrt{3}}.$$

$$\therefore \quad DO^2=AD^2-AO^2=a^2-\frac{a^2}{3}$$

$$=\frac{2a^2}{3}.$$

$$DO = a\sqrt{\tfrac{2}{3}} = \frac{a}{3}\sqrt{6}.$$

The depth below the surface of the liquid of the centre of gravity of a side face then

$$d + \tfrac{1}{3}\,.\,DO = d + \frac{a\sqrt{6}}{9}$$

Therefore thrust on each side face

$$= \frac{1}{2}\,.\,\frac{a^2\sqrt{3}}{2}\left(d + \frac{a\sqrt{6}}{9}\right)w.$$

Thrust on the horizontal face

$$= \frac{1}{2}\,a^2\,.\,\frac{\sqrt{3}}{2}\,dw.$$

If AO meets BC in E, then

$$\tan DEO = \frac{DO}{EO} = \frac{\frac{a\sqrt{6}}{3}}{\frac{a\sqrt{3}}{6}} = 2\sqrt{2}.$$

Hence the resultant thrust on the tetrahedron

$$= 3\left[\frac{a^2\sqrt{3}}{4}\left(d + \frac{a\sqrt{6}}{9}\right)w\,.\,\cos DEO\right] - \frac{a^2\sqrt{3}}{4}\,dw.$$

$$= 3\,.\,\frac{a^2\sqrt{3}}{4}\left(d + \frac{a\sqrt{6}}{9}\right)w\,.\,\frac{1}{3} - \frac{a^2\sqrt{3}}{4}\,dw$$

$$= \frac{a^2\sqrt{3}}{4} \times \frac{a\sqrt{6}}{3}\,w = \frac{a^3\sqrt{2}}{12}\,w.$$

19. Let w be the weight of 1 cubic feet of fresh water. Then the weight of 1 cubic feet of salt water $= 1{\cdot}026\ w$ lbs.

Thrust due to salt water on dock gate $= w'\,.\,z\,.\,s$

$$\text{here } w' = 1{\cdot}026\ w\ ;\ z = \frac{25}{2}$$

$$S = 25 \times 50 \text{ sq. ft.}$$

Hence the thrust due to salt water $= 1{\cdot}026\ w \times \frac{25}{2} \times 25 \times 50.$

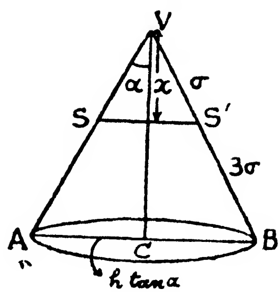
If d is the required depth of fresh water, the thrust due to it is $=50\times d\times\frac{d}{2}\,w$.

Hence from the problem,

$$1{\cdot}026w\times\frac{25}{2}\times 25\times 50=50\,.\,\frac{d^2}{2}\,.\,w$$
$$d^2=25^2\times 1{\cdot}026.$$
$$d=25\sqrt{(1{\cdot}026)}$$
$$=25{\cdot}322\ldots\text{ feet.}$$

20. Let h be the height of the cone and the semi-vertical is α.

Let the common surface SS′ of the liquids be at a distance x from V.



Hence

$$\tfrac{1}{3}\pi x^3\tan^2\alpha=\tfrac{1}{3}\pi\,(h^3-x^3)\tan^2\alpha.$$
$$x^3=h^3-x^3$$
$$2x^3=h^3$$
$$\therefore\qquad x=\frac{h}{2^{1/3}}\quad\ldots\ldots\text{(I)}$$

Let the densities of the liquids be σ and 3σ.

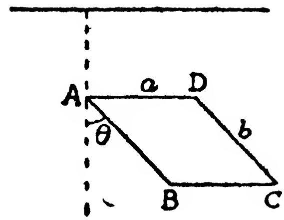
Thrust on the base

$$P_1=\pi h^2\tan^2\alpha\,[x\sigma g+(h-x)\,3\sigma g].$$
$$=\pi h^2\tan^2\alpha\,.\,\sigma g[3h-2x]$$
$$=\pi h^2\tan^2\alpha\,.\,\sigma g\left[3h-\frac{2h}{2^{1/3}}\right]$$
$$=\pi h^3\tan^2\alpha\,.\,\sigma g\,[3-4^{1/3}]$$

When filled with lighter liquid σ.

Thrust, $P_2=\pi h^3\tan^2\alpha\,.\,\sigma g$.

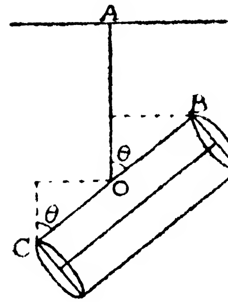
Hence the required ratio $=\frac{P_1}{P_2}=(3-4^{1/3})$.



21. Let w be the weight per unit area of the rectangle $=ab$, depth of the C.G. of the area immersed

$$=c+\frac{b}{2}\cos\theta$$

Hence the required thrust on the rectangle $=w\,.\,ab\left(c+\frac{b}{2}\cos\theta\right)$



22. Let h be the height of the cylinder and a being the radius of the circular ends.

Depth of C.G. of the upper end

$$=c-\frac{h}{2}\cos\theta$$

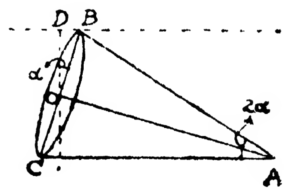
Depth of C.G. of the lower end

$$=c+\frac{h}{2}\cos\theta$$

$$\text{Thrust on the upper end}=\pi a^2\left(c-\frac{h}{2}\cos\theta\right)w$$

$$\text{Thrust on the lower end}=\pi a^2\left(c+\frac{h}{2}\cos\theta\right)w$$

where w is the intrinsic weight.



23. The free surface of the fluid is the horizontal plane passing through B.

Let the radius of the base be n and h the height of the cone.

$$\angle\,\text{BOD}=\alpha$$

$$\text{OB}=h\tan\alpha=r.\quad\therefore\quad h=r\cot\alpha$$

$$\text{OD}=r\cos\alpha.$$

Thrust on the base = Area × depth of C G. × w

$$=\pi r^2\,.\,r\cos\alpha\,.\,w=\pi r^3 w\cos\alpha$$

But the weight of the contained liquid $=\frac{1}{3}\pi r^2 \,.\, r \cot\alpha \,.\, w$

$$=\frac{1}{3}\pi r^3 \,.\, w \,.\, \frac{\cos\alpha}{\sin\alpha}$$

$$\frac{\text{Thrust on the base}}{\text{wt. of the contained liquid}}=3\sin\alpha.$$

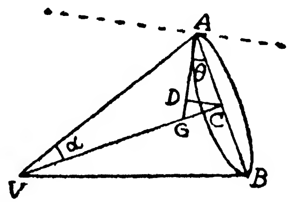
Hence

The thrust on the base $=3\sin\alpha$ (wt. of the contained liquid)

(Hence proved)

24. $VC=h$

$AC=h\tan\alpha$



Let the cone be suspended from a point A on the rim. G is the C.G. of the liquid in the cone.

$$\therefore \quad GC=\frac{h}{4}$$

Let $\angle\, CAD=\theta$

$$\tan\theta=\frac{GC}{AC}=\frac{h}{4\,.\,h\tan\alpha}=\frac{\cos\alpha}{4\sin\alpha}$$

$$\cos\theta=\frac{4\sin\alpha}{\sqrt{(\cos^2\alpha+16\sin^2\alpha)}}=\frac{4\sin\alpha}{\sqrt{(1+15\sin^2\alpha)}}$$

Depth of the C.G. below $A=AD=AC\cos\theta$

$$\therefore \quad AD=h\tan\alpha\,.\,\frac{4\sin\alpha}{\sqrt{(1+15\sin^2\alpha)}}$$

Hence the thrust on the base

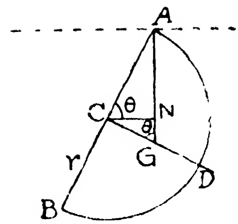
$$=w\,.\,(\pi h^2\tan^2\alpha)\,.\,\frac{h\tan\alpha\,.\,4\sin\alpha}{\sqrt{(1+15\sin^2\alpha)}}$$

$$=\frac{4\pi h^3\, w\,.\,\tan^3\alpha\,.\,\sin\alpha}{\sqrt{(1+15\sin^2\alpha)}}$$

$$\frac{\text{Thrust on the base}}{\text{wt. of the liquid contained}}=\frac{4\pi h^3\, w\,.\,\tan^3\alpha\,.\,\sin\alpha}{\sqrt{(1+15\sin^2\alpha)}}\;\frac{1}{3}\pi h^3\tan^2\alpha\,.\,w$$

$$=\frac{12\,.\,\sin^2\alpha}{\cos\alpha\,.\,\sqrt{(1+15\sin^2\alpha)}} \qquad \text{(Proved)}$$

25. Let the hemisphere be suspended from a point A on the rim of the base.



Let the radius of the hemisphere $=r$

G is the C.G of the hemisphere

$$\therefore \qquad CG=\frac{3r}{8}$$

Wt. of the contained liquid $=\frac{2}{3}\pi r^3 w$.

Thrust on the base $=(\pi r^2) \cdot AN \cdot w$

$$=\pi r^2 \cdot w \cdot r\sin\theta$$

But $$\tan\theta=\frac{AC}{CG}=\frac{r}{\frac{3r}{8}}=\frac{8}{3}$$

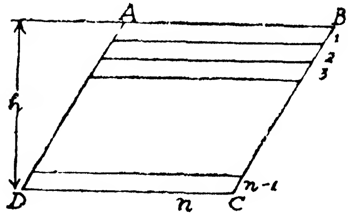
$$\text{Sin }\theta=\frac{8}{\sqrt{(64+9)}}=\frac{8}{\sqrt{(73)}}.$$

$$\therefore \quad \text{Thrust on the base}=\pi r^3 w \cdot \frac{8}{\sqrt{(73)}}.$$

$$\therefore \quad \frac{\text{Thrust on the base}}{\text{wt. of the contained liquid}}=\frac{\pi r^3 w \cdot \frac{8}{\sqrt{(73)}}}{\frac{2}{3} \cdot \pi r^3 w}$$

$$=\frac{12}{\sqrt{(73)}}$$ Proved.

26. Let a parallelogram ABCD be immersed in a homogeneous liquid with the side AB in the surface and let the vertical height of the parallelogram be h. Let $h_1, h_2, \ldots, h_n$ be the heights of the n strips into which the parallelogram is divided. If the pressure on each strip is the same, it is equal to $\frac{1}{n}$ times the thrust on the whole parallelogram.



So if h_r denotes the height of any strip, we must have

$$\frac{1}{n}\cdot w\cdot\frac{h}{2}\cdot a\cdot h=w\cdot ah_r\times(\text{depth of the C.G. of the } r^{th} \text{ strip})$$

w being the intrinsic weight.

$$\frac{h^2}{2n}=h\left(h_1+h_2+h_3+\ldots\ldots+h_{r-1}+\frac{h_r}{2}\right)$$

Let d_r be the depth of the r^{th} dividing line from the surface

$$d_r=h_1+h_2+\ldots\ldots+h_r$$

$$\therefore\quad \frac{h^2}{2n}=h_r\left[\frac{(h_1+h_2+\ldots\ldots+h_r)+(h_1+h_2+\ldots\ldots+h_{r-1})}{2}\right]$$

or $\dfrac{h^2}{n}=(d_r-d_{r-1})\ (d_r+d_{r-1})$

or $\dfrac{h^2}{n}=d^2{}_r-d^2{}_{r-1}$

But we have

$$\frac{1}{n}\ w\cdot\frac{h}{2}\times ah=w\cdot\frac{h_1}{2}\times ah_1 \qquad \text{(first strip)}$$

or
$$h_1=h\cdot\sqrt{\frac{1}{n}}=d_1$$

$$\therefore\quad d_2=\sqrt{\frac{h^2}{n}+d_1{}^2}=\sqrt{\frac{h^2}{n}+\frac{h^2}{n}}=h\sqrt{\frac{2}{n}}$$

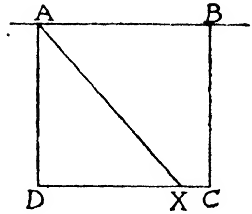
$$d_3=\sqrt{\frac{h^2}{n}+d_2{}^2}=\sqrt{\frac{h^2}{n}+\frac{2h^2}{n}}=h\sqrt{\frac{3}{n}}$$

and so on.

$$dr=h\sqrt{\frac{r}{n}}$$

$$\therefore\qquad dr\propto\sqrt{r}.$$

Hence the depth of the dividing lines are proportional to the square roots of the natural numbers.



27 Let ABCD be the square with side AB in the surface and let the dividing line is AX.

Thrust on $\triangle$ ADX

$=\frac{1}{2}$ (thrust on the square)

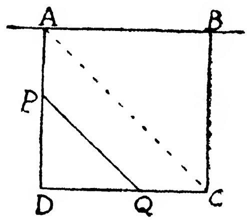
$\therefore\ (\frac{1}{2}\,.\,AD.DX)\,.\,(\frac{2}{3}\ AD)\ w=\frac{1}{2}\,.\,(AD.DC)\times(\frac{1}{2}\ AD)\times w.$

Let $AD=DC=a$

$\frac{1}{3}\,.\,a^2\,.\,D\times w=\frac{1}{4}\,.\,a^3w$

$DX=\frac{3}{4}\,a$

$=\frac{3}{4}\,.\,DC.$



28. Let ABCD be the square, the side AB being in the surface.

Let the reqd. line is PQ, which is parallel to AC.

Let $AP=x$.

$PD=a-x=DQ$

It is given that

Thrust on $\triangle$ PDQ $=\frac{1}{2}$ thrust on the square

$$\frac{1}{2}\,(a-x)\,(a-x)\,.\,[x+\frac{2}{3}\,.\,(a-x)]\ w=\frac{1}{2}\,.\,a^2\,.\left(\frac{a}{2}\right)w$$

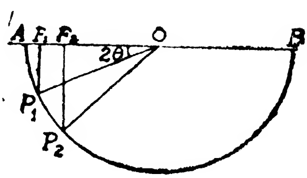
$$(a-x)^2\,(3x+2a-2x)=\frac{3}{2}\ a^3$$

$$(a-x)^2\,.\,(2a+x)=\frac{3}{2}\ a^3$$

$$2\,.\,(a^2+x^2-2ax)\,(2a+x)=3a^3.$$

or $\quad 2x^3-6a^2x+a^3=0$

From this Eqn. we get x.



29 We know that for a sector of angle θ, the area of the sector

$=\frac{1}{2}\ a^2\theta$

and the C. G. of a sector lies at a distance of $\dfrac{2a}{\frac{3\theta}{2}}=\dfrac{4a}{3\theta}$ from the centre on the central radius.

Hence the pressure on the sector AOP_1

$$=a^2\theta \cdot \frac{2a}{3\theta} \cdot \sin\theta \cdot \sin\theta \cdot w$$

$$=\tfrac{2}{3}\, a^3 \cdot \sin^2\theta \cdot w.$$

Pressure on the semi-circle $=\dfrac{\pi a^2}{2} \cdot \left(\dfrac{4a}{3\pi}\right) \cdot w$

$$=\tfrac{2}{3}\, a^3 w.$$

Since pressure on each sector is the same, hence,

$$\frac{2}{3}\, a^3 \cdot \sin^2\theta \cdot w = \frac{1}{n}\left(\frac{2}{3}\, a^3 w\right).$$

$$\sin^2\theta = \frac{1}{n}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{2}{n}.$$

$$OF_1 = a\cos 2\theta = a\left(1 - \frac{2}{n}\right) = a - \frac{2a}{n}.$$

$$AF_1 = OA - OF_1 = a - a + \frac{2a}{n}$$

$$AF_1 = \frac{2a}{n}.$$

Let $\angle P_1OP_2 = 2\varphi$.

Area of the sector $P_1OP_2 = \varphi .\, a^2$

Press on the sector $P_1OP_2 = \varphi \cdot a^2 \cdot \tfrac{2}{3} \cdot \dfrac{\theta \sin\varphi}{\varphi} \cdot \sin(2\theta+\varphi) w$

$$=\tfrac{2}{3}\, a^3 \cdot \sin\varphi .\, \sin(2\theta+\varphi) w,$$

Now accordingly,

$$\tfrac{2}{3}\, a^3 \sin^2\theta \cdot w = \tfrac{2}{3}\, a^3 \cdot \sin\varphi \cdot \sin(2\theta+\varphi)\, w$$

$$a\,(1-\cos 2\theta) = a\,[\cos 2\theta - \cos 2\,(\theta+\varphi)].$$

$$F_1 F_2 = OF_1 - OF_2$$

$$= OP_1 \cos 2\theta - OP_2 \cos(2\theta + 2\varphi)$$

$$= a\cos 2\theta - a\cos(2\theta+2\varphi)$$

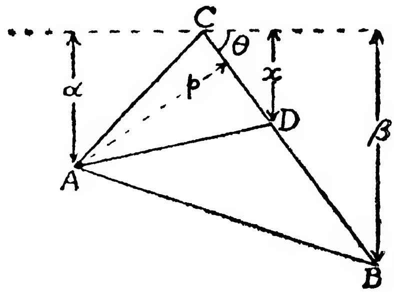
$$= a\,(1-\cos 2\theta)$$

$$=a\left[\ 1-1+\frac{2}{n}\right].$$

$$=\frac{2a}{n}.$$

Divide the horizontal diameter into n equal parts each at a distance equal to $\frac{2a}{n}$, the ordinates at these points will divide the arch of the semi-circle in the required points.

30. Let ABC be the triangle with C in the surface. Through A draw the dividing line AO.



Depth of C. G. of $\triangle$ABC

$=\frac{1}{3}(\alpha+\beta)$

Depth of C. G. of $\triangle$ACD

$=\frac{1}{3}(x+\alpha)$

Area of $\triangle$ ABC$=\frac{1}{2}p\,.\,$BC$=\frac{1}{2}p\,.\,\beta\sec\theta$

Area of $\triangle$ ACD$=\frac{1}{2}p\,.\,$CD$=\frac{1}{2}p\,.\,x\sec\theta$

$\therefore$ Pressure on $\triangle$ ABC$=(\frac{1}{2}p\,.\,\beta\sec\theta)\,\frac{1}{3}(\alpha+\beta)\,w$

$=\frac{1}{6}p\,.\,\beta\,(\alpha+\beta)\sec\theta\,.\,w.$

Pressure on $\triangle$ ACD$=\frac{1}{6}p\,.\,x\,.\,(\alpha+x)\sec\theta\,.\,w.$

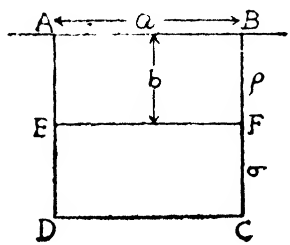
Since the pressure of $\triangle$ ACD is double of $\triangle$ ABC. Therefore,

$$\tfrac{1}{6}p\,\beta\,.\,(\alpha+\beta)\,.\sec\theta\,.\,w=2\,.\,\tfrac{1}{6}px.\,(\alpha\times x)\sec\theta\,.\,w.$$

$$2x\,(\alpha+x)=\beta\,(\alpha+\beta).$$

This equation gives the value for x.

31. Thrust on ABEF



$=ab\,.\,\frac{b}{2}\,.\,\rho g.$

$=\frac{1}{2}ab^2\,\rho g.$

Thrust on EFCD

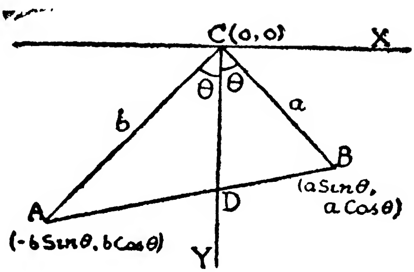
$$=a\,(a-b)\left[\,b\rho g+\frac{a-b}{2}\quad g\,\right]$$

Since the pressures are equal

$$\tfrac{1}{2}\, ab^2\, \rho g = a\,(a-b)\; g\left[b\rho + \left(\frac{a-b}{2}\right)\sigma\right].$$

or $\quad b\,.\,\rho\,(3b-2a)=\sigma\,(a-b)^2.$

32. The figure is self explanatory.



Let CD=Z.

$\triangle ACD=\tfrac{1}{2}\, bz \sin\theta$

$\triangle CBD=\tfrac{1}{2}\, az \sin\theta$

$\triangle ABC=\tfrac{1}{2}\, ab \sin 2\theta$

$\qquad = ab \sin\theta\cos\theta$

Area of $\triangle$ ABC = Area of $\triangle$ ACD + Area of $\triangle$ CBD

$$ab\sin\theta\,.\,\cos\theta=\tfrac{1}{2}\, bz\sin\theta+\tfrac{1}{2}\, az\sin\theta.$$

$$z=\frac{2\,ab\cos\theta}{a+b}$$

$$\text{Pressure on } \triangle \text{ ACD}=\tfrac{1}{2}\, bz\,.\,\sin\theta\,.\,\tfrac{1}{3}\left[b\cos\theta+\frac{2ab\cos\theta}{a+b}\right]w$$

$$\text{Pressure on } \triangle \text{ CBD}=\tfrac{1}{2}\,.\,az\,.\,\sin\theta\,.\,\tfrac{1}{3}\left[a\cos\theta+\frac{2ab\cos\theta}{a+b}\right]w.$$

$$\therefore\quad \frac{\text{Pressure on } \triangle \text{ ACD}}{\text{Pressure on } \triangle \text{ CBD}}=\frac{b}{a}\,.\,\frac{\{b\,(a+b)+2ab\}}{\{a\,(a+b)+2ab\}}$$

$$=\frac{3\,ab^2+b^3}{3\,a^2b+a^3}$$

$\therefore\quad$ Required ratio $=(b^3+3ab^2):(a^3+3a^2b)$

33. Let the thickness of each liquid be a. Hence the side of the box is $n\,a$.

Thrust on the base

$$=(n\,a)^2\, g\,[n\rho\, a+(n-1)\,\rho\,.\,a+\ldots\ldots+2\rho a+\rho a]$$

$$=n^2a^2\, g\,.\,\rho\,.\,a\,[n+(n-1)+\ldots\ldots+2+1]$$

$$=n^2a^2\, g\,.\,\rho a\;\frac{n\,(n+1)}{2}\;\ldots\ldots\ldots\ldots\ldots\; \text{I}$$

Now the liquid above the lower thickness may be replaced by a thickness h of density $n\rho$.

$$x \times n\rho = a\,(n-1)\,\rho + a\,(n-2)\,\rho + \ldots\ldots + a\rho$$

$$x \,.\, n = a\,[(n-1) + (n-2) + \ldots\ldots + a]$$

$$= a\,\frac{n\,(n-1)}{2}$$

$$x = \left(\frac{n-1}{2}\right) a.$$

Hence the thrust on the portion of a side in contact with the lowest liquid

$$= na \,.\, a \,.\, \left[x + \frac{a}{2}\right] . \, n\rho g$$

$$= n^2\,a^2 \,.\, \left[\frac{n-1}{2}\,a + \frac{a}{2}\right] \rho g$$

$$= n^2\,a^2\,g \,.\, \rho \,.\, \frac{n}{2} \quad \ldots\ldots \text{ II}$$

We see that I is $(n+1)$ times of II

34. Thrust on the base of the tumbler

$$= \pi r^2\,g \left[\frac{h\rho}{2} + \frac{h\rho'}{2}\right]$$

$$= \tfrac{1}{2}\pi r^2\,(\rho + \rho')\,hg$$

The whole Pressure is that due to a liquid of density ρ' in contact with the whole curved surface and to one of density $\rho - \rho'$ in contact with the lower half.

$\therefore$ Whole pressure on the curved surface

$$= 2\pi rh \,.\, \frac{h}{2} \,.\, \rho' g + 2\pi r \,.\, \frac{h}{2} \,.\, \frac{h}{4} \,.\, (\rho - \rho')g$$

$$= \tfrac{1}{4} \,.\, \pi\,(3\rho' + \rho)\,rh^2\,g.$$

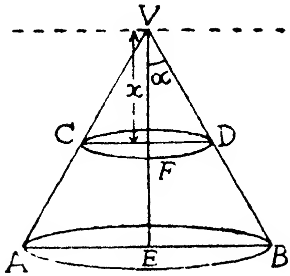
Therefore required ratio

$$= \frac{\text{the pressure on the base of the tumbler}}{\text{whole pressure on its curved surface}}$$

$$=\frac{\frac{1}{3}\pi r^2(\rho+\rho')hg}{\frac{1}{4}\pi(3\rho'+\rho)rh^2g}$$

$$=\frac{2r(\rho'+\rho)}{h(3\rho'+\rho)}$$

35. Let h be the height of the cone and α be the semi-vertical angle, and let x be the depth of the required cutting plane CD.



Area of the curved surface of the cone (V, AB) $=\pi\,.\,(h\tan\alpha)\,.\,h\sec\alpha.$

Depth of the C. G. $=\frac{2}{3}h.$

Thrust on the curved surface of the cone (V, AB)

$$=w\,.\,\pi(h\tan\alpha)\,h\sec\alpha\,.\,\tfrac{2}{3}h.$$

$$=\frac{2\pi}{3}wh^3\tan\alpha\,.\,\sec\alpha.$$

Thrust on the curved surface of the cone (V, CD)

$$=w\,.\,\pi\,.\,(x\tan\alpha)(x'\sec\alpha)\,\tfrac{2}{3}x.$$

$$=\tfrac{2}{3}\pi w\,.\,x^3\,.\,\tan\alpha\,.\,\sec\alpha.$$

It is given,

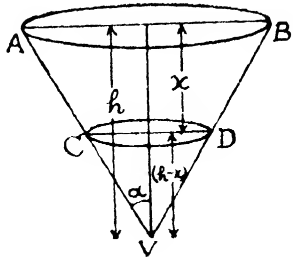
Whole pressure on the upper half = half that on the whole cone.

$$\tfrac{2}{3}\pi wx^3\tan\alpha\,.\,\sec\alpha=\tfrac{1}{2}\left[\frac{2\pi}{3}wh^3\tan\alpha\,.\,\sec\alpha\right]$$

$$x=\frac{h}{2^{1/3}}=\frac{h}{2}\,.\,(4)^{1/3}.$$

II Case

Pressure on the lower half is equal to the same.



$$\therefore\ (h-x)^2\left[x+\frac{(h-x)}{3}\right]=\tfrac{1}{2}h^2-\frac{h}{3}$$

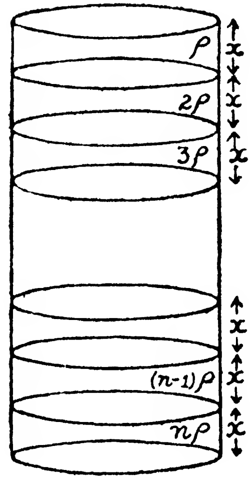
$$4x^3-6hx^2+h^3=0$$

$$(2x-h)(2x^2-4hx-h^2)=0$$

$\therefore\ x=\frac{h}{2}$; $\frac{1+\sqrt{3}}{2}$ x, or $\frac{1-\sqrt{2}}{2}$ x.

$x=\frac{h}{2}$ is the only possible value, since the second $>h$ and the third is negative.

36. Let the thickness of each section of liquid be x. Consider the rth section.



All the sections above it may be replaced by a thickness x of density $r\rho$, then

$$x\ .\ r\rho g=h\ [(r-1)\ \rho g+(r-2)\rho g+\dots\dots\dots+\rho g]$$

$$x\ .\ r=h\ .\ \frac{r\ (r-1)}{2}$$

$$x=(r-1)\ \frac{h}{2}.$$

The whole pressure on this rth section if A is the area of the cross section

$$=\text{A}h\ .\left[x+\frac{h}{2}\right]r\rho g$$

$$=\text{A}\ hr^2\ \rho g$$

i. e., the pressure on rth section varies as r^2. Therefore the pressure on portion in contact with first, second, ... liquid are in the ratio of 1^2, 2^2, ...

(Hence Proved)

EXAMPLES VI

1. Suppose W be the weight of the lid ABCD and a be the side of the box, then

$$\text{W}=\tfrac{2}{3}\ a^3\ w\ \dots\dots\dots\dots\ \text{I}$$

The lid will be on the point of opening if the moment of the weight of the lid about AB is equal to the moment of the thrust on it.

i. e.. $W \times \frac{a}{2} \cos 45 = wa^2 \cdot \frac{a}{2} \sin 45 \times \frac{2}{3} a.$

$$W = wa^2 \cdot \frac{2}{3} a$$

$$= \frac{2}{3} a^3 w$$ (Hence Proved)

2. Let a be the edge of the box.

$\therefore \quad wa^3 = 24$ I

Suppose x is the required depth of water, then the thrust on the side is $axw \cdot \frac{x}{2}$ acting at a distance $\left(a - x + \frac{2x}{3}\right)$.

$$\therefore \quad \frac{x}{2} \cdot axw \left(a - \frac{x}{3}\right) = 5 \cdot \frac{a}{2}.$$

$$\therefore \quad 8x^3 - 24\, ax^2 + 5a^3 = 0.$$

$$\therefore \quad (2x - a)(4x^2 - 10\, ax - 5a^2) = 0$$

$\therefore \quad x = \frac{a}{2}$; other values of x are not possible.

3. Thrust on the lid $= 1.1 \cdot \frac{1}{2} \sin 45 \cdot w$

This thrust acting at a distance $\frac{2}{3}$ ft. from the upper end.

Hence, if W be its weight,

Taking moment about lower edge,

$W \times \frac{1}{2} \cos 45 = 1.1 \cdot \frac{1}{2} \sin 45 \times w \times \frac{1}{2}$

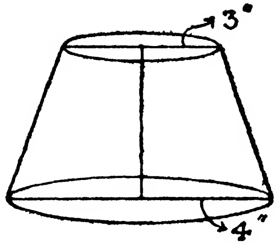
$$W = \frac{1}{3} w = \frac{1}{3} \times \frac{125}{2}$$

$$= 20 \frac{5}{6} \text{ lbs.}$$

EXAMPLES VII

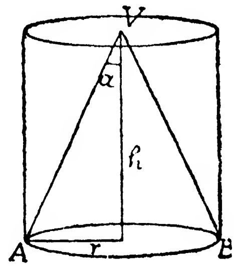
1. The depth in both cases being the same. Hence the thrusts are proportional to the areas *i e.* 16 : 9.

In the first case the pressure of the curved surface on the water has everywhere a downward component. The result-



ant of all these downward components has to be balanced by the thrust in addition to the weight of the water.

In the second case the same pressure has everywhere an upward component which helps to support the weight of the water



2. Suppose α is the semi-vertical angle of the cone and height is h.

Thrust on the table

$$=\pi h^2 \tan^2 \alpha \, . \, hw$$
$$=\pi h^3 \tan^2 \alpha \, . \, w \ldots \text{ I}$$

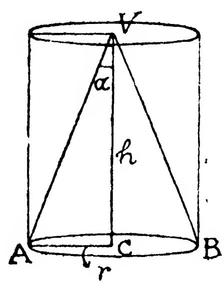
Resultant thrust of water upon the glass

$=$weight of the superincumbent liquid

$=(\pi h^2 \tan^2 \alpha \, . \, h - \frac{1}{3} \pi h^3 \tan^2 \alpha) \, . \, w$

$=\frac{2}{3} \pi h^3 \tan^2 \alpha \, . \, w$

$=\frac{2}{3}$ (Thrust on the table) (Proved)

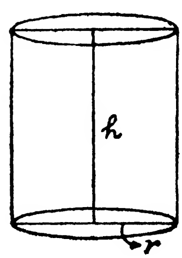


3. Thrust on the base of the cone of height h and radius r

$=$weight of the super-incumbent liquid

$=$weight of the cylinder of water of height h and of the same base $=\pi r^2 hw \ldots$ I

Volume of the water in the cone will be $=\frac{1}{3} \pi r^2 hw$. This poured in the cylinder of radius r. Let it occupied the height h,



$$\therefore \qquad \pi r^2 h_1 = \frac{1}{3} \pi r^2 h$$

$$h_1 = \frac{h}{3}$$

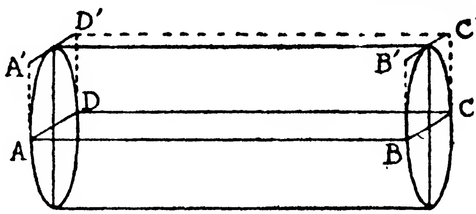
Thrust on the base of the cylinder of height $\frac{1}{3} h$ and radius r

$$=\pi r^2 \, . \, \tfrac{1}{3} hw \ldots \text{II}$$

$$\frac{\text{Thrust on the base of the cone}}{\text{Thrust on the base of the cylinder}}=\frac{\pi r^2 hw}{\frac{1}{3}\pi r^2 hw}=\frac{3}{1}=3:1$$

(Proved)

4. Let ABCD be the horizontal plane through the axis which divides the cylinder into two parts.



Vertical thrust on the lower half of the cylinder

$=$ weight of the super-incumbent liquid

$=$[volume of the lower half of the cylinder + volume of the Prism on ABCD as base] . w

$=[\frac{1}{2}\pi r^2 h+$ area of the rectangle ABCD $\times$ AA′] w

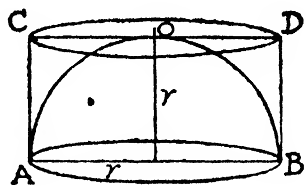
$=[\frac{1}{2}\pi r^2 h+2r\,.\,h\,.\,r]\,w$

$$=\left[\frac{\pi}{2}+2\right]r^2hw.$$

5. Thrust on the table

$=\pi r^2\,.\,r\,.\,w$

$=\pi r^3 w$ I



Through each point of the circular edge of the base of the hemisphere draw a vertical line to meet the horizontal plane through the highest point of the hemisphere. Then the thrust equals the weight of the water that could be included between the surface thus formed and the hemisphere, and hence

The resultant vertical thrust on the surface

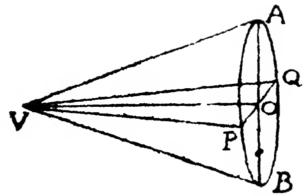
$=(\pi r^3 w-\frac{2}{3}\pi r^3 w)$

$=\frac{1}{3}\pi r^3 w$.........II

$=\frac{1}{3}$ (thrust on the table).

6. Let h be the height and r the radius of the base of the cone (V,AB), with its axis horizontal.

Vertical thrust on the upper half of the curved surface



=the weight of the super-incumbent liquid

=(volume of the prism on base VPQ

$-\frac{1}{2}$ volume of the cone) w

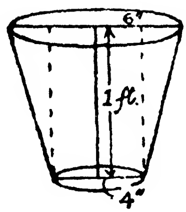
$=(\frac{1}{2} \cdot 2r \cdot h \cdot r = \frac{1}{2} \cdot \frac{1}{3}\pi r^2 h)\, w$

$=\left(1-\frac{\pi}{6}\right) r^2 h\, w$

Similarly, vertical thrust on the lower half

$$=\left(1+\frac{\pi}{6}\right) r^2 h\, w$$

7. Vertical thrust



=weight of frustum of water

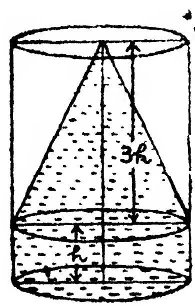
−wt. of cylinder, radius $\frac{1}{3}$ ft. and height 1 foot.

$$=\left[\frac{\pi}{3}\left(\frac{1}{3^2}+\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{2^2}\right) - \pi \cdot \frac{1}{3^2} \cdot 1\right] w$$

$$=\left(\frac{19\pi}{108}-\frac{\pi}{9}\right)\frac{1000}{16} \text{ lbs. wt.}$$

$$=\frac{875}{216}\pi \text{ lbs. wt.}$$

8. Vertical thrust on the curved surface of the cone acting upward



=weight of the super-incumbent liquid

$=(\pi r^2 \cdot 3h - \frac{1}{3}\pi r^3 \cdot 3h)\, w$

$=2\, \pi r^2 hw.$

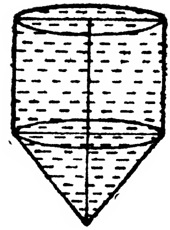
Inverted position.

Vertical thrust

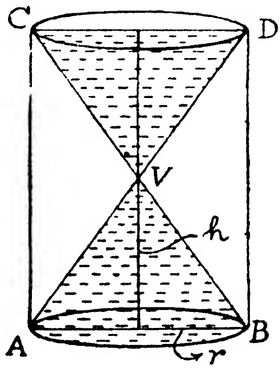
$=$ weight of the contained liquid

$=(\pi r^2 h+\frac{1}{3}\,\pi r^2\,.\,3h)\,w$

$=2\,\pi r^2\,hw.$



(Hence Proved)



9. (V, AB) and (V, CD) are two equal cones with radius r and height h.

Vertical thrust on the lower cone

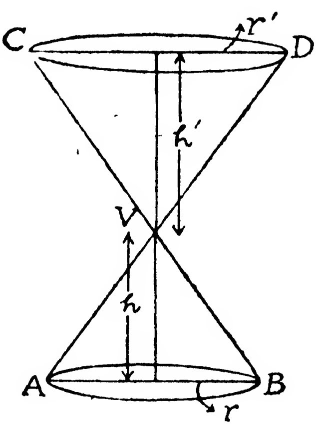
$=$ weight of the super-incumbent liquid.

$=\pi r^2\,.\,2\,hw-\frac{1}{3}\,\pi r^2\,.\,hw$

$=\frac{5}{3}\,\pi r^2\,hw$

$=\frac{5}{2}\,.\,(\frac{2}{3}\,\pi r^2\,hw)$

$=\frac{5}{2}$ times the weight of water contained.



10. Let h and r be the height and radius of the base of the lower cone, h' and r' those of the upper.

Vertical thrust of the curved surface of the upper cone acting downwards

$=$ wt. of the liquid contained

$=\frac{1}{3}\,\pi r'^2\,h'\,.\,w$ ......

Vertical thrust on the curved surface of the lower cone acting upwards

$=$ weight of the super-incumbent liquid

$=\pi r^2\,(h+h')\,w-\frac{1}{3}\,\pi r^2\,hw$

$=\frac{1}{3}\,\pi r^2 w\,(3h+3h'-h)=\frac{1}{3}\,\pi r^2\,w\,(2h+3h')$

Hence the resultant vertical thrust

$$=\frac{1}{3}\pi r^2 w\,(2h+3h')-\frac{1}{3}\pi r'^2\,h'w$$

Since $\frac{h}{r}=\frac{h'}{r'}$ $\therefore$ $r'=\frac{h'}{h}\cdot r$ and $r'^2=\frac{h'^2}{h^2}\cdot r^2$

Putting this value of r', we get

Resultant vertical thrust

$$=\frac{1}{3}\pi r^2 w\left(3h'+2h-\frac{h'^3}{h^2}\right)$$

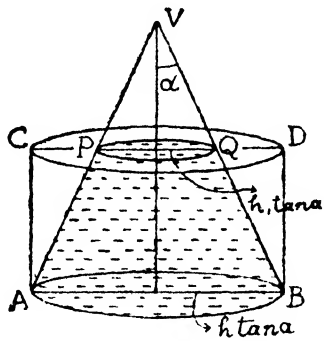
and this should be zero when $2h=h'$

$$3h'+2h-\frac{h'^3}{h^2}=0,$$

which is clearly zero.

Hence Proved

11. Let h be the height, α the semi-vertical angle of the cone (V, AB). Also, let h_1 be the distance of the water surface PQ from V.



Vertical thrust on the curved surface of the cone

=the weight of the super-incumbent liquid.

$$=\pi r^2\,(h-h_1)\,w-\frac{1}{3}\pi\,(h^3-h_1^3)\tan^2\alpha w$$

$$=\pi h^2\,(h-h_1)\tan^2\alpha-\frac{1}{3}\pi\,(h^3-h_1^3)\tan^2\alpha.$$

Weight of the cup

$$=\frac{5}{8}\cdot\frac{1}{3}\pi h^3\tan^2\alpha w.$$

The cup will be on the point of rising, when

$$w\pi h^2\,(h-h_1)\tan^2\alpha-\frac{1}{3}\pi\,(h^3-h_1^3)\tan^2\alpha\cdot w=\frac{5}{24}\pi h^3\tan^2\alpha\cdot w,$$

$$h^2\,(h-h_1)-\frac{1}{3}\,(h^3-h_1^3)=\frac{5}{24}\,h^3.$$

$$24h^3-24h^2h_1-8h^3+8h_1^3-5h^3=0.$$

$$11h^3-24h^2h_1+8h_1^3=0.$$

or $(h-2h_1)\,(11h^2-2h\,h_1-4h_1^2)=0.$

either $h-2h_1=0$

i. e. $h_1=\frac{h}{2}$

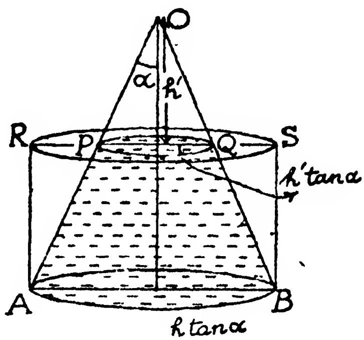
or $11h^2-2h\,h_1-4h_1^2=0$

i. e. $\dfrac{h}{h_1}=\dfrac{2\pm\sqrt{210}}{22}$

But the later value is inadmissible for h is not less than h_1.

$$\therefore \quad h_1=\frac{h}{2}.$$

12. Let h be the height, α the semi-vertical angle of the cone (O, AB). Let h' be the distance of the water surface PQ from the vertex O of the cone.



The resultant vertical thrust on the curved surface

=weight of the superincumbent liquid.

=weight of the liquid in the cylinder RABS whose base is AB and height $(h-h')$—the weight of the liquid in PABQ.

$=[\pi r^2 (h-h')-\frac{1}{3}\pi (h^3-h'^3) \tan^2 \alpha]\, w$

Weight of the cone$=\frac{1}{3}\pi (h^3-h'^3) \tan^2 \alpha \,.\, w$.

Cone will be on the point of rising when

$\pi h^2 (h-h') \tan^2 \alpha \,.\, w-\frac{1}{3}\pi (h^3-h'^3) \tan^2\alpha \,.\, w$
$=\frac{1}{3}\pi (h^3-h'^3) \tan^2\alpha \,.\, w$

$h^2 (h-h')-\frac{1}{3} (h^3-h'^3)=\frac{1}{3} (h^3-h'^3)$.

$h^2 (h-h')-\frac{2}{3} (h^3-h'^3)=0$

$(h-h') [\frac{2}{3} (h^2+h'^2+hh')-h^2]=0$

$(h-h') [2h'^2-h^2+2hh']=0$

either $h-h'=0$, but h/h'

or $2h'^2-h^2+2hh'=0$

$$h'=\frac{-2h\pm\sqrt{4h^2+8h^2}}{4}=\frac{-h\pm\sqrt{3}\, h}{2}$$

$$=\frac{\sqrt{3}-1}{2}\, h.$$

Since the other values being inadmissible.

$$W=\tfrac{1}{3}\pi\left[h^3-\frac{h^3}{8}(3\sqrt{3}-9+3\sqrt{3}-1)\right]\tan^2\alpha w$$

$$=\tfrac{1}{3}\pi\, h^3\tan^2\alpha\,.\,w\,.\,\tfrac{1}{8}\,[8-3\sqrt{3}+9-3\sqrt{3}+1]$$

$$=\tfrac{1}{3}\,.\,\pi\, h^3\tan^2\alpha\,.\,w\,.\,\tfrac{1}{4}\,.\,(9-3\sqrt{3}).$$

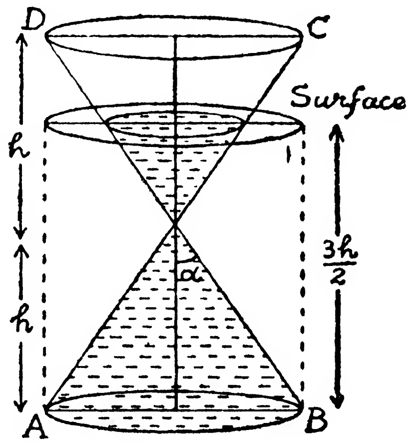
Weight of the liquid reqd. to fill the cone

$$=\tfrac{1}{3}\,\pi h^3\tan^2\alpha\,.\,w:$$

Hence reqd. ratio $=\dfrac{9-3\sqrt{3}}{4}$.

Proved

13 Let h be the height of either cone, α the semi-vertical angle and W the weight of any one of them. The liquid (of intrinsic wt. w, say) will be on the point of escaping between the lower cone and the table when the vessel (double cone) is at the point of rising upwards. In this position the resultant upward vertical thrust upon the vessel (say F) must balance its lower weight 2W. Now the vertical thrust (F_1) on the lower cone is



= Weight of the super-incumbent liquid

= Weight of a cylinder of water of height $\frac{3}{2}h$ and radius $h\tan\alpha$ — weight of the liquid in the lower cone

$$=\pi h^2\tan^2\alpha\,.\,\frac{3\,hw}{2}\,w-\tfrac{1}{3}\,\pi h^3\tan^2\alpha w$$

$$=\tfrac{7}{6}\,\pi h^3\tan^2\alpha w.$$

Similarly vertical thrust on the curved surface of the upper cone which is in contact with the liquid is (F_2)

$=$Weight of the super-incumbent liquid

$=$Weight of the liquid contained in the upper cone

$$=\frac{1}{3}\pi\,.\,\frac{h^3}{8}\tan^2\alpha\,w=\frac{\pi}{24}h^3\tan^2\alpha\,.\,w.$$

In the position of limiting equilibrium, $F=2W$

Also weight of the liquid which either cone can hold is

$$W'=\frac{1}{3}\pi h^3\tan^2\alpha\,.\,w$$

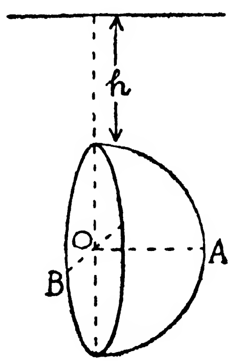
$$\therefore\quad \text{Required ratio}=\frac{W}{W'}=\frac{\frac{F}{2}}{W'}$$

$$=\frac{\frac{27}{48}\pi h^3\tan^2\alpha\,.\,w}{\frac{1}{3}\pi h^3\tan^2\alpha\,.\,w}$$

$$=\frac{27}{16}.$$

Proved

EXAMPEES VIII



1. Horizontal thrust on curved surface $=$ pressure on circular end

$X=w\,.\,\pi a^2 h.$ I

Vertical thrust

$=$Weight of water displaced by the hemisphere

$Y=\frac{2}{3}\pi a^3\,.\,w$ II

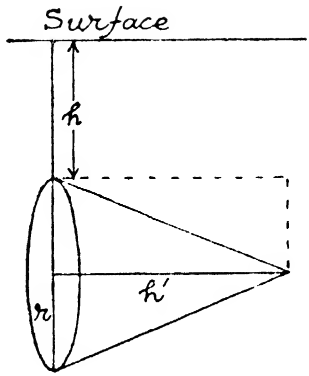
If R is the resultant thrust making an angle θ with the horizontal

$$R=\sqrt{(x^2+y^2)}=\pi a^2 w\left(h^2+\frac{4a^2}{9}\right).$$

$$\tan\theta = \frac{Y}{X} = \frac{\frac{2}{3}\pi a^3 w}{w \cdot \pi a^2 h} = \frac{2a}{3h}.$$

$$\therefore \quad \theta = \tan^{-1}\frac{2a}{3h}.$$

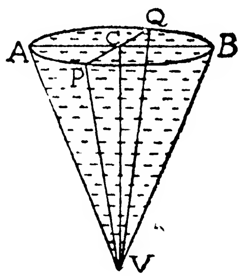
Proved



2. Let the radius of the base is r and height h'. Resultant horizontal thrust in a direction perpendicular to the paper,

$= (rh') \cdot h \cdot w.$

$= r\,hh'\,w.$

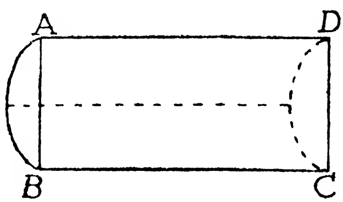


3. Horizontal thrust on half-curved surface VAPQ = Pressure on the projection of this on the vertical plane VPQ

= whole pressure on $\triangle$ VPQ.

$= w \cdot \frac{1}{2} \cdot 2r \cdot h \cdot \frac{h}{3}$

$= \frac{1}{3}\, rh^2 w$



4. Let r be the radius and h the length of the cylinder. Let the intrinsic weight of the liquid be w.

This cylinder is cut off by a vertical plane through the axis.

Horizontal thrust in a direction perp. to the rectangular face ABCD of half the cylinder

= Pressure on ABCD

$= 2r \cdot h \cdot r \cdot w = 2r^2 hw$

Vertical thrust=weight of the contained liquid

$$=\frac{\pi r^2 h}{2} w.$$

Resultant thrust$=\frac{1}{2} r^2hw \cdot \sqrt{(\pi^2+16)}$

$$=r^2hw\sqrt{4+\frac{\pi^2}{4}}$$

line of action made with the horizontal an angle

$$\tan\theta=\frac{\pi}{4}$$

with the vertical an angle $\tan^{-1}\frac{4}{\pi}$

5. Let h be the height and r the radius of the base of the cylinder.

Vertical thrust$=\frac{1}{2}\pi r^2 \cdot \frac{h}{2} \cdot (w+2w)$

$$=\frac{3}{4}\pi r^2hw$$

Horizontal thrust=thrust on the vertical rectangular section

$$=2rh \cdot \frac{h}{2} w+2r \cdot \frac{h}{2} \cdot \frac{h}{4} \cdot w$$

$$=rh^2w\,(1+\tfrac{1}{4})$$

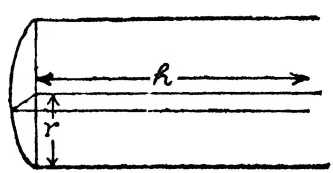
$$=\tfrac{5}{4}\,rh^2w$$

If the resultant thrust making an angle θ with the horizon

$$\tan\theta=\frac{\frac{3\pi}{4}r^2hw}{\frac{5}{4}\cdot rh^2w}=\frac{3\pi r}{5h}$$

$$\theta=\tan^{-1}\frac{3\pi r}{5h}$$

6. If h is the length and r the radius of the cross-section of the pipe, then, W the weight of liquid which fills half the pipe



$$W=\left(\frac{\pi r^2 h}{2}\right) w$$

$$\frac{W}{\pi}=\frac{hr^2}{2}\, w \,.........\mathrm{I}$$

Horizontal thrust on either curved surface

$=$thrust on the rectangle of length h and radius r

$$=hr\,.\,\frac{r}{2}\,.\,w$$

$$=\frac{hr^2}{2}\,w=\frac{W}{\pi}\,......\mathrm{II}$$

Vertical thrust on either half

$=$weight of the contained water

$$=\frac{\pi r^2 hw}{4}$$

$$=\frac{W}{2}\,......\mathrm{III}$$

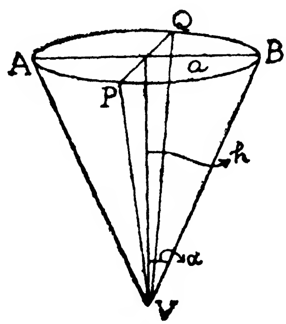
If θ is the angle that the resultant thrust makes with verti cal

$$\cot\theta=\frac{\text{Vertical thrust}}{\text{Horizontal thrust}}$$

$$=\frac{\frac{W}{2}}{\frac{W}{\pi}}=\frac{\pi}{2}$$

$$\therefore\qquad \theta=\cot^{-1}\left(\frac{\pi}{2}\right)$$

7. Let a and h be the radius and height of the cone, w be the intrinsic weight of water. When the cone has been divided by a vertical plane through the axis, the vertical thrust



$=$weight of the liquid displaced by the half-cone

$=\frac{1}{6}\,\pi a^2 hw$ I

and it acts vertically upwards through the C.G. of the semi-cone.

The horizontal component in a direction perp. to the paper is zero, for the semi-cone is symm. about the vertical plane.

Horizontal thrust

$$=\tfrac{1}{2}(2a \,.\, h)\frac{h}{3} \,.\, w$$

$$=\frac{ah^2}{3}w \;\ldots\ldots\text{II}$$

Resultant thrust $=\tfrac{1}{6}\,ahw\,\sqrt{(\pi^2a^2+4h^2)}$

If the resultant thrust makes an angle θ with the horizontal, then

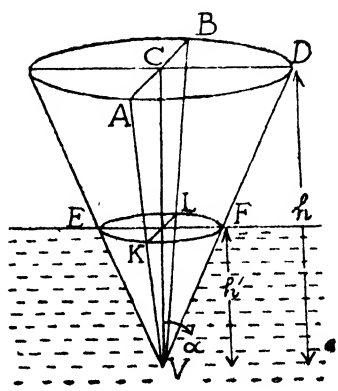
$$\tan\theta=\frac{\tfrac{1}{6}\pi a^2 hw}{\frac{ah^2}{3}w}$$

$$=\frac{\pi a}{2h} \qquad \text{But } \frac{a}{h}=\tan\alpha$$

$$=\frac{\pi\tan\alpha}{2}$$

$$\therefore \qquad \theta=\tan^{-1}\left(\frac{\pi\tan\alpha}{2}\right)$$

8. Let the plane VAB divide the cone into two equal parts. It floats with its height h' in the liquid. The two parts are hinged at V.



If σ be the specific gravity of the cone.

the weight of the cone

$$=\tfrac{1}{3}\pi h^3\tan^2\alpha \,.\, \sigma w$$

Upward thrust $\tfrac{1}{3}\pi h'^3\tan^2\alpha w$ must balance their weight.

$$\therefore \qquad \tfrac{1}{3}\pi h'^3 \tan^2 \alpha w = \tfrac{1}{3}\pi\ h^3 \tan^2 \alpha\ \sigma\ w$$
$$h'^3 = h^3\sigma \qquad \text{.........I}$$

The centre of gravity of the base of the cone is at a distance $\frac{4a}{3\pi}$ from the centre.

$\therefore$ distarce of the C.G. of the half cone

$$= \frac{3}{4} \cdot \left(\frac{4\ .\ a}{3\pi}\right) = \frac{3}{4} \cdot \frac{4h \tan \alpha}{3\pi} = \frac{h \tan \alpha}{\pi}$$

Similarly the C G. of one part of the cone immersed in water *i e.* of liquid displaced $= \frac{h' \tan \alpha}{\pi}$

Horizontal thrust on half the curved surface

$$= \frac{2r'\ .\ h'}{2} \cdot \frac{h'}{3} w = \frac{h'^3 \tan \alpha}{3} w$$

and acts at the centre of pressure of the tiangle *i e.* at a distance of $\frac{h'}{2}$ from the vertex.

The weight of half the cone $= \frac{1}{6}\ \pi h^3 \tan^2 \alpha\sigma w$

$$= \tfrac{1}{6}\ \pi h'^3 \tan^2\ \alpha w \text{ from I}$$

Now the two parts will not separate if the sum of the moments of the horizontal thrusts, wt. of the liquid displaced by half the cone is greater than the moment of the weight of half the cone about the hinge at the vertex.

Therefore, we have

$$\frac{h'^3 \tan \alpha}{3} w \cdot \frac{h'}{2} + \frac{\pi\ h'^3\ \tan^2 \alpha}{6} w \cdot \frac{h' \tan \alpha}{\pi}$$
$$> \tfrac{1}{6}\ \pi\ h'^3 \tan^2 \alpha\ .\ w \frac{h \tan \alpha}{\pi}$$

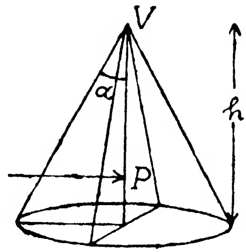
or $$\frac{h'}{6} + \frac{\pi\ h' \tan^2 \alpha}{6\pi} . > \frac{\pi h \tan^2\alpha}{6\pi} .$$

$$\frac{h'}{6}\left(1 + \tan^2 \alpha\right) > \frac{h}{6} \tan^2 \alpha.$$

$$h' \sec^2 \alpha > h \cdot \frac{\sin^2 \alpha}{\cos^2 \alpha}.$$

or $$h' > h \sin^2 \alpha.$$

9. Water flow out when separated.



Thrust acting downward on the base

$$= \frac{1}{2} \pi h^2 \tan^2 \alpha \cdot hw .$$

$$= \frac{1}{2} \pi h^3 \tan^2 \alpha \cdot w.$$

Vertical thrust on the curved surface acting upward

$$= \left(\frac{1}{2} \pi h^3 \tan^2 \alpha - \frac{1}{3} \cdot \frac{1}{2} \pi h^3 \tan^2 \alpha \right) w$$

$$= \frac{1}{3} \cdot \pi h^3 \tan^2 \alpha \cdot w.$$

$\therefore$ Resultant thrust in the vertical direction

$$= \frac{1}{6} \pi h^3 \tan^2 \alpha \cdot w.$$

= Weight of the water contained

Horizontal thrust $= \frac{1}{2} \cdot 2h \tan \alpha \cdot h \cdot \frac{2}{3} hw$

$$= \frac{2}{3} h^3 \tan \alpha \cdot w$$

it will act through the centre of pressure at P.

$$VP = \frac{3}{4} h.$$

The water will not flow out if the moment of the vertical thrust about the hinge is greater than that of the horizontai thrust.

Hence the water will not flow out if

$$\frac{1}{6} \pi h^3 \tan^2 \alpha \cdot w \cdot \frac{h \tan \alpha}{\pi} \geqslant \frac{2}{3} h^3 \tan \alpha \cdot w \cdot \frac{3}{4} h$$

$$\text{or} \quad \tan^2 \alpha \geqslant 3$$

$$\tan \alpha \geqslant \sqrt{3}$$

$$\therefore \quad \alpha \geqslant 60$$

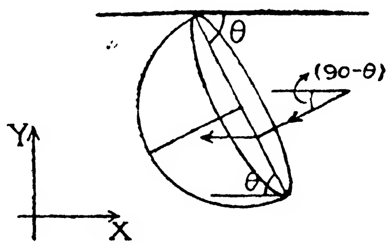
When vertical angle is greater than 120°, water will not flow out.

EXAMPLES IX

1. It is given

$$\tan \theta = 2 \; ; \sin \theta = \frac{2}{\sqrt{5}}, \; \cos \theta = \frac{1}{\sqrt{5}}.$$

Let the components of the resultant thrust on the curved surface be X and Y in the horizontal and vertical directions respectively.



By Archimedes' principle upward thrust on the solid = weight of water displaced

$$= \frac{2}{3} \pi a^3 w$$

when w is the intrinsic weight of the liquid.

Thrust on the plane base

$$= \pi a^2 \,.\, a \sin \theta \,.\, w$$

$$= \pi a^3 . \frac{2}{\sqrt{5}} w$$

$$= \frac{2}{\sqrt{5}} \pi a^3 w$$

Hence resolving horizontally and vertically, we have

$$Y - \frac{2}{\sqrt{5}} \pi a^3 w \cos \theta = \frac{2}{3} \pi a^3 w$$

$$Y = \frac{2}{3} \pi a^3 w + \frac{2}{\sqrt{5}} \,.\, \pi a^3 w \,.\, \frac{1}{\sqrt{5}}$$

$$= \frac{16}{15} \pi a^3 w \;\ldots\ldots\ldots\ldots\ldots\ldots \; \text{I}$$

$$X=+\frac{2}{\sqrt{5}}\,\pi a^3\,w\sin\theta=\frac{2}{\sqrt{5}}\,.\,\pi a^3\,w\,.\,\frac{2}{\sqrt{5}}$$

$$=\frac{4}{5}\,\pi a^3\,w \;\ldots\ldots\ldots\ldots\ldots \;\text{II}$$

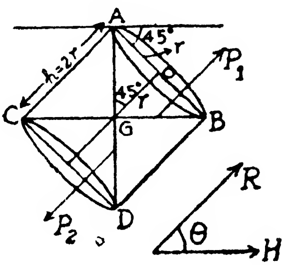
Resultant thrust on the curved surface, $R=\sqrt{(X^2+Y^2)}$

$$\therefore\quad R=\tfrac{4}{5}\,\pi a^3\,w\,.\,\sqrt{\tfrac{16}{9}+1}=\tfrac{4}{3}\,\pi a^3\,w$$

$$=2\,(\tfrac{2}{3}\pi a^3\,w)$$

$=2$ (weight of the displaced liquid).

2. In the equilibrium position AG should be vertical, and, therefore, the axis would make an angle of 45° with the vertical. Since $OA=OG=x$ and $\angle AOG=90°$.



Thrust on the upper face AB is

$$P_1=\pi r^2\,.\,r\sin 45\,.\,w$$

$$=\frac{\pi r^3\,w}{\sqrt{2}}$$

Thrust on the lower face CD is

$$P_2=\pi r^2\,(r+2r)\sin 45°\,w$$

$$=\frac{3\,\pi r^3\,w}{\sqrt{2}}$$

Now suppose that the resultant thrust on the curved surface of the cylinder is R and it makes an angle φ with the horizontal. Then,

$R\cos\varphi+P_1\cos 45=P_2\cos 45$ (resolving horizontally)

and $-(R\sin\varphi+P_1\sin 45)+P_2\sin 45=$ Total resultant

$=$ Vertical thrust on the cylinder

$=$ Weight of water contained

$=\pi r^2\,.\,2r\,.\,w=2\pi r^3 w.$

This gives

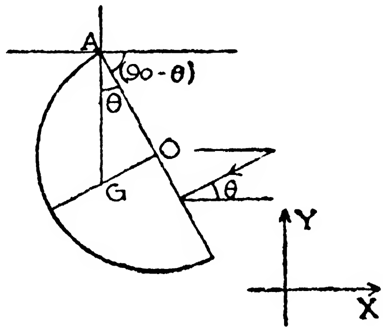
$$R \cos \varphi = (P_2 - P_1) \cos 45 = \frac{2\pi r^3 w}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \pi r^3 w$$

and $R \sin \varphi = -2\, \pi r^3 w + (P_2 - P_1) \sin 45°$.

$$= -2\pi r^3 w + \frac{2\pi r^3 w}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = -\pi r^3 \text{ w.}$$

which shows, horizontal and vertical comp. of the resultant thrust is equal to half the weight of the contained water.

3. If A be the highest point of the rim, O the centre of the plane face and G be the centre of gravity, the AG must be vertical,



$$\tan \theta = \frac{OG}{AO} = \frac{3}{8}$$

Thrust on the plane end

$$= \pi a^2 \cdot a \cos \theta \cdot \text{w.}$$

$$= \pi a^3 \cdot \frac{8}{\sqrt{73}} \text{ w.}$$

Thrust on the hemisphere = weight of the contained water.

$$= \frac{2}{3} \pi a^3 \text{w.}$$

$$X = \pi a^3 \cdot \frac{8}{\sqrt{73}} \cdot \text{w} \cdot \cos \theta = \pi a^3 \cdot \frac{8}{\sqrt{73}} \cdot \text{w} \cdot \frac{8}{\sqrt{73}}$$

$$= \frac{64}{73} \cdot \pi a^3 \text{ w}$$

$$Y = \pi a^3 \cdot \frac{8}{\sqrt{73}} \text{ w} \cdot \sin \theta + \frac{2}{3} \pi a^3 \text{w}$$

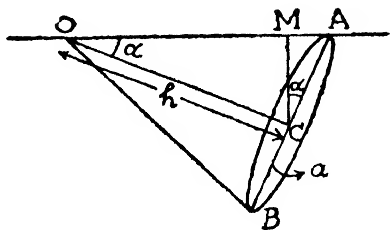
$$= \pi a^3 \cdot \text{w} \left[\frac{24}{73} + \frac{2}{3}\right]$$

$$=\pi a^3 w\left[\frac{72+146}{73.3}\right]=\frac{218}{73.3}\pi a^3 w$$

$$\tan\varphi=\frac{Y}{X}=\frac{218}{73.3}\times\frac{73}{64}=\frac{109}{96}.$$

$$\varphi=\tan^{-1}\frac{109}{96}.$$

4. Let 2α be the vertical angle of the cone, the inclination of the plane base to the vertical is α, and hence the thrust X on the plane base



$X=\pi a^2 \cdot a\cos\alpha \cdot w$

Hence required horizontal thrust on curved surface

$$=X\cos\alpha=\pi a^3 w\cdot\cos^2\alpha.$$

Vertical thrust$-$X sin α

=downward vertical thrust on the whole cone

=weight of the contained water

$$=\frac{1}{3}\pi a^2 hw=W$$

Hence the required vertical thrust

$$=\frac{\pi}{3}a^2 hw+\pi a^3 w\sin\alpha\cdot\cos\alpha.$$

$$=W+3W\cdot\frac{a\sin\alpha\cdot\cos\alpha}{h} \quad \text{but} \quad \frac{a}{h}=\tan\alpha$$

$$=W+3W\cdot\sin\alpha\cdot\cos\alpha\cdot\tan\alpha.$$

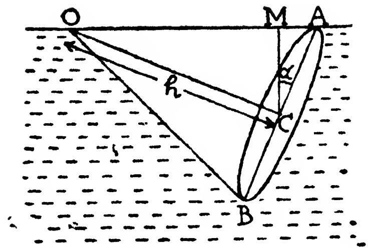
$$=W(1+3\sin^2\alpha)$$

Horizontal thrust$=\pi a^3 w\cdot\cos^2\alpha.$

$$=3w\cdot\cos^2\alpha\,\frac{a}{h}$$

$$=3w\cos^2\alpha\cdot\tan\alpha.$$

$$=3w\cdot\sin\alpha\cdot\cos\alpha.$$



5. Let the generating line OA is in the surface. r, the radius of the base and α is the semi-vertical angle of the cone.

Resultant vertical thrust V, on the whole cone

$$=\text{the weight of water displaced}$$

$$=\frac{1}{3}\pi r^2 h \,.\, w$$

Thrust on the circular base

$$X=\pi r^2 \,.\, r \,.\, \cos\alpha \,.\, w$$

$$=\pi r^3 \cos\alpha \,.\, w.$$

Hence vertical thrust on curved surface

$$=V-X\sin\alpha.$$

And horizontal thrust$=X\cos\alpha$.

If θ be the angle to the vertical of the resultant thrust on the curved surface, thence,

$$\tan\theta=\frac{X\cos\alpha}{V-X\sin\alpha},$$

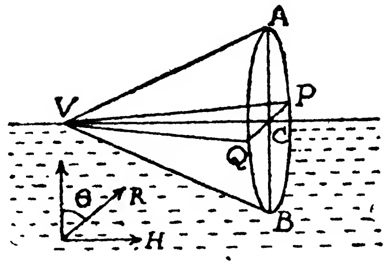
$$=\frac{\pi r^3 w \cos\alpha\cos\alpha}{\frac{1}{3}\pi r^2 h \,.\, w-\pi r^3 w\cos\alpha \,.\, \sin\alpha}$$

$$=\frac{r \,.\, \cos^2\alpha}{\frac{h}{3}-r\cos\alpha \,.\, \sin\alpha}=\frac{3h \,.\, \tan\alpha \,.\, \cos^2\alpha}{h-3\tan\alpha \,.\, \cos\alpha \,.\, \sin\alpha}$$

$$=\frac{3\sin\alpha \,.\, \cos\alpha}{1-3\sin^2\alpha}=\frac{3\sin\alpha \,.\, \cos\alpha}{\cos^2\alpha+\sin^2\alpha-3\sin^2\alpha}$$

$$=\frac{3\sin\alpha \,.\, \cos\alpha \,.}{\cos^2\alpha-2\sin^2\alpha}=\frac{3\tan\alpha \,.}{1-2\tan^2\alpha},$$

$$\theta=\tan^{-1}\frac{3\tan\alpha}{1-2\tan^2\alpha}.$$



6. Let (V_1 AB) be the cone of height h and radius r. The cone floats with its axis in the surface and half its volume immersed.

Resultant vertical thrust V

=weight of the liquid displaced

$$=\frac{1}{6}\pi r^2 \,.\, hw$$

Thrust X, on the semi-circular base

$$=\frac{1}{2}\pi r^2 \cdot \frac{4r}{3\pi} \cdot w$$

$$=\frac{2}{3} r^3 w.$$

If R is the resultant thrust on the curved surface acting at an angle θ to the vertical, then resolving horizontally and vertically, we get

$$R \cos\theta=\frac{1}{6}\pi r^2 hw.$$

$$R \sin\theta=\frac{2}{3} r^3 w.$$

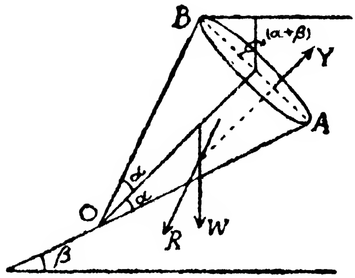
$$\tan\theta=\frac{\frac{2}{3} r^3 w}{\frac{1}{6}\pi r^2 hw}$$

$$\tan\theta=\frac{4}{\pi}\cdot\frac{r}{h}.$$

$$\text{Since } \frac{r}{h}=\tan\alpha.$$

$$\therefore \quad \tan\theta=\frac{4}{\pi}\,.\tan\alpha$$

7. Let the semi-vertical angle of the cone is α, which is filled with liquid be placed on the inclined plane at an angle β to the horizon.



Resultant vertical thrust V_1 of the water

$=$weight of the water

$=\frac{1}{3}\pi h^3 \tan^2 \alpha \,.\, w$

AB will make an angle $(\alpha+\beta)$ with the vertical.

Thrust on the plane base *i.e.* X

$$=(\pi h^2 \tan^2 \alpha)\, r \cos(\alpha+\beta) \,.\, w$$

If R is the thrust on the curved surface making an angle θ with the horizontal,

$$R \sin\theta - X \sin(\alpha+\beta) = V$$
$$R \cos\theta = X \cos(\alpha+\beta)$$

$\therefore$ R sin θ (vertical component)

$=\frac{1}{3}\pi h^3 \tan^2 \alpha \,.\, w + \pi h^2 \tan^2 \alpha \,.\, h \tan\alpha \times \cos(\alpha+\beta) \,.\, \sin(\alpha+\beta).$

$=[w + 3w \,.\, \tan\alpha \,.\, \sin(\alpha+\beta) \,.\, \cos(\alpha+\beta)]$

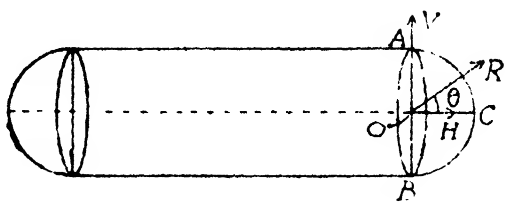
$=W\,[1+\frac{3}{2} \,.\, \tan\alpha \,.\, \sin 2(\alpha+\beta)]$

R cos θ (Horizontal component)

$=\pi h^2 \tan^2 \alpha \,.\, h \tan\alpha \,.\, \cos(\alpha+\beta)\cos(\alpha+\beta) \,.\, w$

$=3w \,.\, \tan\alpha \,.\, \cos^2(\alpha+\beta)$

8. Horizontal thrust in a direction perp. to OC is zero by symmetry. Horizontal thrust in the direction OC



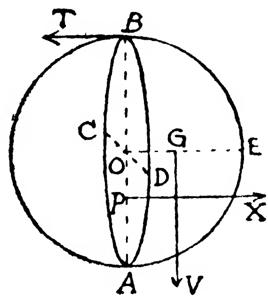
$=$whole pressure on the projection of the curved surface on the vertical plane.

$$=\pi a^2 \,.\, aw = \pi a^3 w = H.$$

Vertical thrust $V = \frac{2}{3}\pi a^3 w.$

$\therefore$ Resultant thrust, $R=\sqrt{(H^2+V^2)}=\frac{\sqrt{13}}{3}\pi a^3 w$ makes an angle $\theta=\tan^{-1}\frac{V}{H}=\tan^{-1}\frac{2}{3}$ to the horizontal and passes through the centre of the hemispherical end.

9. Let a be the radius of the sphere. Let AOB be the diameter of the sphere dividing the sphere into two parts which are hinged at the lowest point A.



Horizontal thrust in the direction CD will be zero by symmetry.

Horizontal thrust in the direction OE

$$=\pi a^2 . a . w$$
$$=\pi a^3 w.$$

acting through P, the centre of pressure of the circle AOBD, such that

$$BP=a+\frac{a^2}{4a}=\frac{5a}{4}$$

and
$$AP=2a-\frac{5a}{4}=\frac{3}{4}a.$$

Vertical thrust on the curved surface V

$$=\text{weight of the contained liquid}$$
$$=\tfrac{2}{3}\pi a^3 w$$

acting through the C.G. of the liquid contained

$$i.e.\ C.G.=\frac{3a}{8}.$$

If T be the tension of the string at highest point B, then taking moment about B, we have

$$T.AB-X.AP-V.OG=0.$$

$$T.2a-X.\frac{3a}{4}-V.\frac{3a}{8}=0.$$

$$T \,.\, 2a = \pi\, a^3\, w \,.\, \frac{3a}{4} + \tfrac{2}{3} \,.\, \pi\, a^3 w \,.\, \frac{3a}{8}$$

$$= \pi\, a^3\, w \,.\, \frac{3a}{4}\,(1+\tfrac{1}{3}).$$

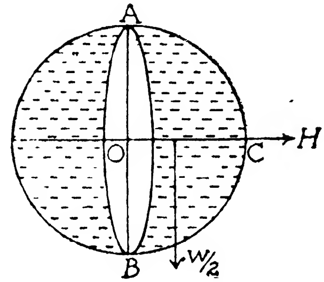
$$= \pi\, a^3\, w \,.\, a$$

$$T = \frac{\pi a^3 w}{2}$$

$$= \tfrac{3}{8} \,.\, (\tfrac{4}{3}\, \pi a^3 w)$$

$= \frac{3}{8}$ (weight of the water contained by the sphere).

10. The horizontal thrust on the hemispherical shell ACB in a horizontal direction perp. to OC will be zero because of the complete symm. of the curved surface of this portion about the vertical plane ABC.



Also, the horizontal thrust in the direction OC

= whole pressure on the projection of the curved surface on a vertical plane through A

$= \pi r^2 \,.\, r \,.\, w = \pi r^3 w$

$= \frac{3}{4}$ w′, where w′ is the weight of contained water

The two hollow hemisphere will remain in contact if the moment of the weight $\left(\frac{W}{2}\right)$ of one of the hemispheres about the hinge A acting in the clockwise direction (tending to keep the two portions together) exceeds that of the resultant horizontal and vertical thrusts on this which tends to take them apart.

But the moment of the vertical thrust V passing through O about A is zero. Hence the necessary condition is

$$\frac{W}{2} \cdot \frac{r}{2} > \tfrac{3}{4}\, W' \,.\, r$$

$$W > 3\, W'$$

EXAMPLES X

1. Let the volume of the man is V.

Weight of the water displaced

$$=\left(V-\frac{4}{12\times 12\times 12}\right)\times w$$

But the condition that the man may float is

$$\left(V-\frac{4}{1728}\right)w=160$$

$$\left(V-\frac{1}{432}\right)\cdot\frac{125}{2}=160$$

$$\left(V-\frac{1}{432}\right)\frac{25}{2}=32$$

$$V-\frac{1}{432}=\frac{64}{25}$$

$$V=\frac{64}{25}+\frac{1}{432}$$

$$\therefore \quad V=2\frac{6073}{10800}$$

2. Let V and V′ are the volumes of the iron and cork,

$$V'\cdot\tfrac{1}{4}\times w=1$$

Also,

$$V\times 7\times w+V'\cdot\tfrac{1}{4}\cdot w=(V+V')\cdot 1\cdot w$$

$$7\,Vw+1=(V+V')\,w$$

$$6\,Vw=wV'-1=4-1$$

$$6\,Vw=3$$

$$Vw=\tfrac{1}{2}$$

$\therefore$ Weight of iron $=V\cdot 7w=\frac{7}{2}=3\frac{1}{2}$ lbs.

3. Let V be the volume of the body; its specific gravity is one since it just floats in water.

Therefore,

$$V\cdot 1+42{\cdot}5=V\times 1{\cdot}85$$

$$\therefore \quad V=\frac{42{\cdot}5}{{\cdot}85}$$

$$=50 \text{ cubic cms.}$$

4. Suppose σ=specific gravity of the gas compared with air

It is given, $w = 1{\cdot}2$ oz

Therefore,

$$\tfrac{3}{2}\,\sigma\,.\,w + 1\text{ oz} = \tfrac{3}{2}\,.\,1\,.\,w$$

$$\therefore \quad \tfrac{3}{2}\,\sigma\,.\,\tfrac{6}{5} = \tfrac{3}{2}\,.\,\tfrac{6}{5} - 1 = \tfrac{4}{5}$$

$$\therefore \quad \sigma = \tfrac{4}{9} = \dot{4}$$

$$\frac{\text{Weight of a cubic foot of the gas}}{\text{Weight of a cubic foot of air}} = \frac{\sigma \times 1{\cdot}2}{1000} = \frac{8}{15000} = {\cdot}0005\dot{3}.$$

5. Let V being the volume in cubic decimetres

Therefore,

$$V \times {\cdot}089 + 50000 = V \times 1{\cdot}2$$

$$\therefore \quad V = \frac{50000}{1{\cdot}111}$$

$$\therefore \quad \text{Volume} = \frac{50000}{1{\cdot}111} \text{ cubic decimetres.}$$

$$= \frac{50}{1{\cdot}111} \text{ cubic metres.}$$

$$= 45\,\frac{5}{1111} \text{ cubic metres}$$

6. Let V is the volume of the iron and σ being the specific gravity, we get

$$275 = \frac{5}{9}\,V \times 15{\cdot}59$$

$$\therefore \quad V = \frac{275 \times 9}{5 \times 15{\cdot}59}$$

$$= 31\,\frac{1171}{1559}$$

Also $\quad 275 = V\,\sigma w$

Since $w = 1$

$$\therefore \quad \sigma = \frac{275}{V}$$

$$= \frac{275 \times 5 \times 15{\cdot}59}{275 \times 9}$$

$$=\frac{5}{9}\times 15{\cdot}59$$

$$=8{\cdot}66\dot{1}.$$

7. Let x feet be the depth below the surface of the water, we get

$$x\times 1{\cdot}025=(x+30)\times{\cdot}918$$

$$x\,(1{\cdot}025-{\cdot}918)=30\times{\cdot}918$$

$$x=\frac{27{\cdot}54}{{\cdot}107}=257\,\frac{41}{107}\text{ feet.}$$

8. Let V be the volume of the ship below the water line and x feet the ship rises when goes to sea water.

$$V\,.\,1\,.\,w=1000\times 2240$$

$$=(V-15000\,x)\times 1{\cdot}026\,w$$

Hence $[1000\times 2240-w\ 15000\,x]\ 1{\cdot}026=1000\times 2240$

$$\therefore\quad x=\frac{1000\times 2240\times{\cdot}026}{15000\times w\times 1{\cdot}026}$$

$$={\cdot}726\ \ldots\ldots\ \text{inch.}$$

9. Let W be the weight of the ship in tons, and V being the volume immersed below the water level. Also, let the area of cross-section of the ship near water is A.

Let W tons be the weight of cubic inch of water

$$\text{Weight per cubic inch of sea-water}=\frac{41}{40}\,w.$$

for Equilibrium of sea-water

$$W=V\,.\,\frac{41}{40}\,w\ \ldots\ldots\ldots\ldots\ \text{I}$$

for Equilibrium in river-water

$$W=(V+a\,.\,A)\,w\ \ldots\ldots\ \text{II}$$

After discharging x tons of cargo

$$W-x=[V+(a-b)\,A]\,w\ \ldots\ldots\ldots\ \text{III}$$

Subtract III from (2), we get

$$x=bAw$$

$\therefore \quad A=\dfrac{x}{bw}$ IV

Equating W from I and II

$$(V+a\ A)\ w=V\ .\ \frac{41}{40}\ w.$$

$$\frac{V}{40}=A\ .\ a=\frac{ax}{bw}$$

$$\therefore \quad V=\frac{40\ ax}{bw}$$

$$W=\frac{40\ ax}{bw}\ .\ \frac{41}{40}\ w=\frac{41\ ax}{b}$$

10. Let x oz be the snow fallen.

Originally length immersed$=\cdot 81=\frac{4}{5}$ feet.

Condition for the equilibrium after the snow has fallen

$$\frac{4}{5}\times 1\cdot 025\ w=\frac{4}{5}\times 1\ .\ w+x.$$

$$x=\frac{4}{5}\ .\ (1\cdot 025-1)\ w=\frac{4\times\cdot 025}{5}\times 1000=20\ \text{oz.}$$

11. Suppose V is the volume of the pomegranate wood and V′ is the volume of the other portion.

From the condition of floating bodies

$$V\times 1\cdot 35\ w+V'\times\cdot 65\times w=(V+V')\ .\ 1\ .\ w$$

or $\quad V\ (1\cdot 35-1)=V'\ (1-\cdot 65)$

$$\therefore \quad V=V'$$

Hence the volume of the two portions of wood are equal

12. Let the specific gravity of the cork is σ and V, V′ are the two volumes

Therefore, $\quad V\ \sigma w=19$ and $V'\times 10\cdot 5\times w=63$

Also, $\quad 19+63=(V+V')\ .\ 1\ .\ w.$

$$\therefore \quad \frac{19}{\sigma}+\frac{63}{10\ 5}=82$$

$$\therefore \quad \sigma=\frac{1}{4}.$$

13. Let the length of the iron portion is l inches.

Then from the condition of the flo·ting bodies

$$l\times 7{\cdot}5\ w+2\times 21\ w=(x+1)\ .\ 13{\cdot}5\ .\ w$$

or $\quad l.\ (13{\cdot}5-7{\cdot}5)=42-13{\cdot}5.$

or $\quad 6l=28{\cdot}5$

$\therefore \quad l=4{\cdot}75.$

14. Let V be the apparent volume of the gold and V′ the volume of the cavity.

Therefore, $\quad (V-V')\times 19{\cdot}25=96{\cdot}25 \ldots\ldots\ldots \mathrm{I}$

and $\quad V\times 1=6 \quad \ldots\ldots\ldots \mathrm{II}$

$\therefore \quad V-V'=5$

$\quad V=6$

$\therefore \quad V'=1$ cubic cm.

15. Let V is the volume of the man and V′ is the volume of .he cork. Therefore,

$$V\times 1{\cdot}1\times w=140 \ldots\ldots\ldots\ldots \mathrm{I}$$

and from the condition of floating bodies

$$V\times 1{\cdot}1\times w+V'\times{\cdot}24\times w=(V+V')\ .\ 1\ .\ w \ldots\ldots\ldots \mathrm{II}$$

from I, $\quad Vw=\frac{140}{1{\cdot}1}=\frac{1400}{11}$

from II, $\quad V'\times{\cdot}76\ w={\cdot}1\ Vw$

$\therefore \quad V'=\frac{10}{76}\ V=\frac{10}{76}\times\frac{1400\times 2}{11\times 125}.$

$$=\frac{112}{418}=463\ \frac{1}{209}\ \text{cubic inches.}$$

16. Let l be the length and $4r$ the external radius of the pencil and σ is the specific gravity of the lead.

Therefore,

$$\pi r^2 l\times\sigma w+\pi\ (16r^2-r^2)\ l\times{\cdot}78\times w=\pi\ .\ 16r^2\ .\ \frac{7l}{8}\ .\ 1.w$$

$$\therefore \quad \sigma + 15 \times \cdot 78 = 16 \cdot \frac{7}{8}$$

$$\sigma = 14 - 15 \times \cdot 78 = 2 \cdot 3.$$

17. Let σ be the specific gravity of the wood.

Then clearly, specific gravity of the two liquids will be $\frac{5}{4}\sigma$ and $\frac{3}{2}\sigma$.

When the liquids are mixed in equal quantities by weight, the specific gravity of the mixture is given by

$$= \frac{W+W}{\frac{W}{\frac{5}{4}\sigma} + \frac{W}{\frac{3}{2}\sigma}} = \frac{15}{11}\sigma$$

If x is the required fraction of volume immersed

$$1 \times \sigma = x \frac{15\sigma}{11} \qquad \therefore \quad x = \frac{11}{15}.$$

18. Let m be the mass of the solid, V its total volume and ρ_1, ρ_2, ρ_3 the densities of the three fluids. According to the principle of Archimedes the mass of a solid floating in any liquid is equal to the mass of the liquid displaced and so we have

$$\frac{V}{a}\rho_1 = \frac{V}{b} \cdot \rho_2 = \frac{V}{c} \cdot \rho_3 = m \ldots\ldots\ldots\ldots \text{I}$$

Case I. When equal volumes, say V, of the three fluids are mixed together, we have the mean density

$$\frac{\text{Total mass}}{\text{Total volume}} = \frac{V\rho_1 + V\rho_2 + V\rho_3}{V+V+V} = \frac{\rho_1 + \rho_2 + \rho_3}{3}$$

Hence the volume of the solid immersed in the fluid while floating on it is

$$\frac{m}{\frac{\rho_1+\rho_2+\rho_3}{3}} = \frac{3}{\frac{\rho_1}{m} + \frac{\rho_2}{m} + \frac{\rho_3}{m}} = \frac{3}{\frac{a}{V} + \frac{b}{V} + \frac{c}{V}}$$

$$= \frac{3}{a+b+c} \cdot V.$$

Case II. When equal weights, say M, of each fluid are mixed together, the mean density

$$=\frac{3M}{\frac{M}{\rho_1}+\frac{M}{\rho_2}+\frac{M}{\rho_3}}=\frac{3}{\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}}$$

Hence the volume of the solid immersed in the mixture i. e. the volume displaced

$$\frac{m}{\frac{3}{\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}}}=\frac{\frac{m}{\rho_1}+\frac{m}{\rho_2}+\frac{m}{\rho_3}}{3}=\frac{\frac{V}{a}+\frac{V}{b}+\frac{V}{c}}{3}$$

$$=\frac{1}{3}\left[\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right]V.$$

19. Let x be the mass of iron which is attached to the bottom. Therefore,

$$x-\frac{x}{7\cdot5}=26.$$

$$x\left(1-\frac{1}{7\cdot5}\right)=26.$$

$$x=\frac{7\cdot5\times26}{6\cdot5}=30 \text{ lbs.}$$

20. Let x inch be the height of water that has to be poured in order that the level inside and outside is the same. Then the externally immersed height is $(x+1)$ inch.

$\therefore$ Weight of water displaced

$$=12\times12\times(x+1)\times\frac{62\cdot5}{12^3} \text{ lbs.}$$

Again, weight of the box + the water inside is

$$\left\{\frac{15\times62\cdot5}{4\times12}+10\times10\times x\times\frac{62\cdot5}{12^3}\right\} \text{ lbs.}$$

$\therefore$ By Archimedes principle,

$$\frac{12\times12\times(x+1)\times62\cdot5}{12^3}=\frac{15\times62\cdot5}{4\times12}+10.10\,.\,x\ \frac{62\cdot5}{12^3}$$

or $144\,(x+1)=15\cdot36+100\,x.$

$44\ x=540-144=396.$

$\therefore\qquad x=9''.$

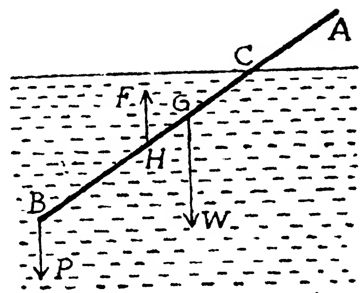
Hence the volume of water that has been poured in is

$10\times10\times9=900$ cubic inch

$\therefore$ Externally immersed depth$=9+1$

$=10$ inch.

21. Let AB be the rod of length $2a$, density ρ and cross-section α. It floats in an inclined position with a length $CA=\dfrac{2a}{n}$ out of water. The resultant thrust F of water acts from H, the middle point of BC. Also since the upward thrust = the weight of water displaced.



$$\therefore\qquad F=2a\left(1-\frac{1}{n}\right)\alpha\,.\,g$$

Resolving vertically

$$2a\left(1-\frac{1}{n}\right)\alpha\,.\,g=P+W\ldots\ldots\ldots\ldots\ldots I$$

Taking moment about, we get

$$2a\left(1-\frac{1}{n}\right)\alpha g\times a\left(1-\frac{1}{n}\right)=W\,.\,a\ldots\ldots\ldots II$$

From II

$$2a\left(1-\frac{1}{n}\right)\alpha g=\frac{nW}{n-1}$$

Substituting in I,

$$\frac{nW}{n-1}=P+W$$

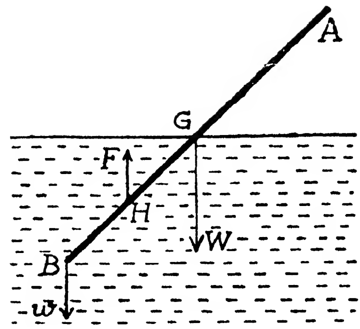
$$(n-1)\ P=W.$$

22. From the previous question,

$$n=2$$

$$\therefore \qquad W=(2-1)\ P$$

$$W=P.$$



23. Let AB be the rod floating with half of its portion BG inside water. If $2l$ be the length of the rod, α its cross-section and ρ the density, then

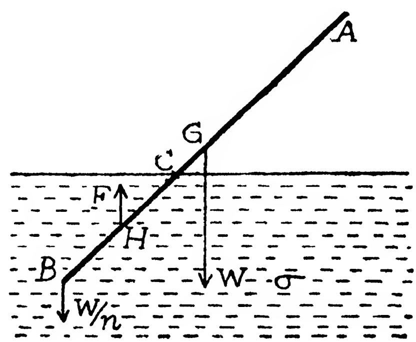
Weight $W=2\,.\,l\,.\,\alpha\rho g$. Also the resultant thrust F of water acts from the middle point H of BG and is equal to the weight of the displaced water.

$$\therefore \quad F=l\,.\,\alpha\,.\,1\,.\,g$$

A heavy metal weight W is attached at B. Taking moment about B for the equilibrium of the rod, we have

$$l\,.\,\alpha\,.\,g\,.\,\frac{l}{2}=2l\,.\,\alpha\,.\,\rho\,.\,g\,.\,l$$

$$\rho=\tfrac{1}{4}.$$



24. Let $2a$ be the length of the rod, α its cross-section, ρ the density and x the length BC that floats inside liquid. The resultant thrust F of the liquid will act upwards from the middle point H of BC.

Now clearly

$$F=\frac{W}{n}+W$$

or $$F=\left(1+\frac{1}{n}\right)W \;\ldots\ldots\ldots\ldots \text{I}$$

Taking moment about B, we have

$$F\cdot\frac{x}{2}=W\cdot a \quad\text{or}\quad F=\frac{2aW}{x}$$

Substituting in I

$$\therefore \quad x=\frac{2an}{n+1}$$

Now $W=2a\,.\,\alpha\,.\,\rho g$ and $F=x\alpha\,.\,\sigma g$

$$\therefore \quad \frac{F}{W}=\frac{x\sigma}{2a\rho}=\frac{2a\,n\sigma}{2a\rho\,(n+1)} \;\ldots\ldots\ldots\ldots \text{II}$$

Also from I, $$\frac{F}{W}=\frac{n+1}{n} \;\ldots\ldots\ldots\ldots \text{III}$$

Comparing II and III

$$\frac{n\sigma}{\rho\,(n+1)}=\frac{n+1}{n}$$

$$\rho\,.\,(n+1)^2=n^2\,\sigma.$$

Hence the rod will float in any position if the above condition is satisfied.

25. The bottle descends the air is compressed and occupies less and less volume so that the water displaced decreases and therefore the upward thrust diminishes. Finally there will be a position where the upward thrust of the water just equals the weight of the bottle, and here it will neither sink or rise. If immersed to a greater depth than this position it will sink.

26. Let A square feet be the area of the water-section of the ship, so that

$$A\,.\,\tfrac{1}{12}\,w=30\times2240 \;\ldots\ldots\ldots\ldots \text{I}$$

Let W be the weight of the ship and V the original volume displaced, so that

$$W = V \cdot w \ldots\ldots\ldots \text{II}$$

and $W - 600 \times 2240 = (V - 2A) \times w' = (V - 2A) . 64 \ldots \text{III}$

Substituting in (3) from I and II, we have

$$W - 600 \times 2240 = \frac{64 \times 2}{125} W - 2 \times 64 \times \frac{30 \times 2240 \times 12 \times 2}{125}$$

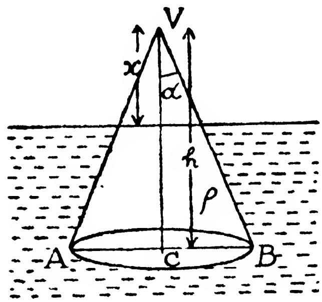
$$\therefore \quad \frac{3W}{125} = \left[\frac{4 \times 64 \times 30 \times 2240 \times 12}{125} - 600 \times 2240 \right] \text{lbs.}$$

$$\therefore \quad W = \left[\frac{4 \times 64 \times 30 \times 12}{3} - 200 \times 125 \right] \text{tons}$$

$$= 5720 \text{ tons.}$$

27. If x be the length out of the liquid

Hence from Archimedes principle

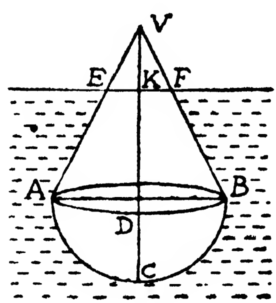


$$\pi h^2 \tan^2 \alpha . h . \rho g$$
$$= \pi (h^3 - x^3) \tan^2 \alpha . \sigma g$$
$$h^3 \rho = (h^3 - x^3) \sigma$$
$$x^3 \sigma = h^3 . (\sigma - \rho).$$
$$x^3 = h^3 \left(1 - \frac{\rho}{\sigma}\right)$$
$$x = h \left(1 - \frac{\rho}{\sigma}\right)^{1/3}$$

28. VD = 7 inch

AB = 2 inch

VK = 3 inch.



Volume of the cone $= \frac{1}{3} \pi . 1^2 . 7$

$$= \frac{7}{3} \pi$$

Volume of the portion of the cone out of liquid

$$=\left(\frac{3}{7}\right)^3\times\frac{7\pi}{3}=\frac{9\pi}{49}.$$

Volume of the hemisphere $=\frac{2}{3}\pi$

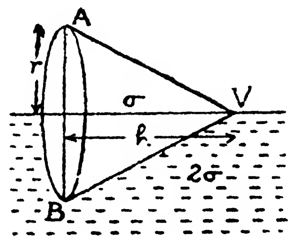
From the principle of Archimedes

$$\left(\frac{7\pi}{3}+\frac{2\pi}{3}-\frac{9\pi}{49}\right)\sigma g=\left(\frac{2\pi}{3}\times\frac{7}{4}+\frac{7\pi}{3}\times\frac{3}{2}\right)g$$

$$\left(3-\frac{9}{49}\right)\sigma=\frac{7}{6}+\frac{7}{2}$$

$$\therefore\quad \sigma=\frac{343}{207}.$$

29. Let the density of the cone is σ and that of liquid 2σ.



Since the axis of the cone is in the surface of the liquid, then half the volume of the cone is in liquid and hence the upward thrust of the liquid just balances the weight of the cone. Hence

$$\frac{1}{2}\ \pi r^2h\,(2\sigma)=\pi r^2h\ .\ \sigma$$

i. e. $\quad \pi r^2h\ .\ \sigma=\pi r^2h\ .\ \sigma\ .$

Proved

30. Let h be the height, α the semi-vertical angle and x the depth originally immersed, and w the weight of the cone.

Since cone is hollow, its weight $=\pi h^2\tan\alpha\ .\ \sec\alpha\ .\ w.$

Weight of the displaced water $=\frac{1}{3}\ \pi r^3\tan^2\alpha\ .\ w\ .$

When water is filled, the total weight of the vessel

$$=\pi h^2\tan\alpha\ .\ \sec\alpha\ .\ w+\frac{1}{3}\ \pi x^3\tan^2\alpha\ .\ w$$

And then weight of the displaced water

$$=\frac{1}{3}\pi h^3 \tan^2\alpha \, . \, w \, .$$

Therefore,

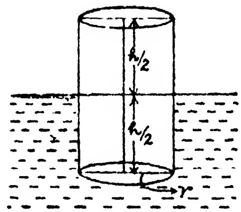
$$\pi x^2 \tan\alpha \, . \, \sec\alpha \, . \, w+\frac{1}{3}\pi x^3 \tan^2\alpha \, . \, w=\frac{1}{3}\pi h^3 \tan^2\alpha \, . \, w.$$

or $\quad 2x^3=h^3$

$$x=\frac{h}{2^{1/3}} \, .$$

EXAMPLES XI

1. Let the specific gravity of the cylinder is σ



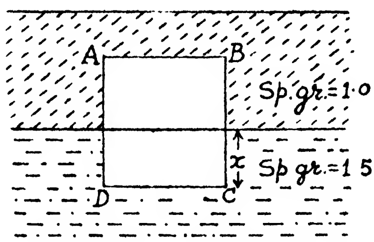
Therefore,

$$\pi r^2 h\sigma=\pi r^2 \frac{h}{2} \, . \, 1+\pi r^2 \, . \, \frac{h}{2} \, . \, (\cdot 0013)$$

$$\therefore \qquad \sigma=\frac{1\cdot 0013}{2}$$

$$=\cdot 50065$$

2. ABCD is the cross-section of the cube



Let x be the depth in the lower liquid

Hence

$$1\times 1\cdot 2=x\times 1\cdot 5+(1-x)\cdot 1$$

$$\cdot 2=\cdot 5x$$

$$\therefore \qquad x=\tfrac{2}{5}.$$

3. Let σ is the specific gravity of Mercury. From the condition of floating bodies

$$5\cdot 1432\sigma=5\cdot 0697\sigma+1\times 1$$

$$\sigma=\frac{1}{\cdot 0735}$$

$$=13\cdot 6054$$

4. Let V and V′ are the volumes of gold and silver. From the principle of Archimedes

$$V \times 19{\cdot}25w + V' \times 10{\cdot}5 \times w = \frac{15}{16}(V+V') \times 13{\cdot}6w$$
$$+\frac{1}{16}(V+V') \times 1 \times w$$

or $$V\left[19{\cdot}25 - \frac{15}{16} \times 13{\cdot}6 - \frac{1}{16}\right] = V'\left[\frac{15}{16} \times 13{\cdot}6 + \frac{1}{16} - 10{\cdot}5\right]$$

i.e. $$103V = 37V'$$

Hence the required ratio

$$= \frac{V \times 19{\cdot}25}{V' \times 10{\cdot}5} = \frac{407}{618}$$

5. Let A be the cross-section of the block and h the depth of its lower surface below water's free surface in the first case.

Then for equilibrium

$$A \times 40 \times {\cdot}9 \times w = A \times h \times w$$

$\therefore$ $$h = 36 \text{ cm.}$$

In the second case, let the oil of specific gravity 0·6 be poured to a height x so that the depth of the lower face of the block below water-oil surface of separation becomes $40-x$. Applying the fundamental condition for equilibrium, we get

$$A\,.\,x \times 0{\cdot}6 + A(40-x) = A \times 40 \times 0{\cdot}9$$
$${\cdot}4x = 40 - 36 = 4.$$
$$x = 10 \text{ cm.}$$

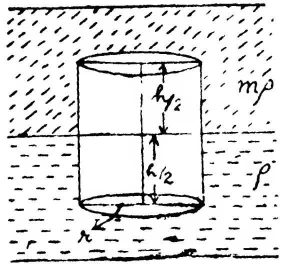
$\therefore$ $$(40-x) = 30 \text{ cm.}$$

Hence $$h-(40-x) = 36-30 = 6 \text{ cm.}$$

i.e. the block will rise through 6 cm.

6. For the equilibrium of the body, the weights of the displaced liquid and displaced air must be equal to the weight of the body. When the air is removed, more liquid must be displaced and hence the body will sink.

7. The density of the cylinder is $n\rho$.



From the principle of Archimedes

$$\pi r^2 h\, n\rho = \pi r^2 \frac{h}{2}\, \rho + \pi r^2\, \frac{h}{2}\, .\, m\rho$$

$$\therefore \qquad n = \frac{m+1}{2}$$

8. The first condition gives specific gravity of the body$=\frac{1}{2}$

Let x be the fraction of the body in water

$$1 \times \tfrac{1}{2} \quad x \times 1 + (1-x)\,.\,80 \times \cdot 00125$$

$$\therefore \qquad x = \tfrac{4}{9}$$

9. Specific gravity of cube$=\frac{4}{5}$.

Let x be the new fraction immersed,

$$x \times 1 + (1-x) \times \cdot 013 = 1 \times \tfrac{4}{5} = \cdot 8$$

$$x \times \cdot 987 = \cdot 789$$

$$x = \frac{789}{987}$$

$$\therefore \qquad \frac{\text{new fraction}}{\text{old fraction}} = \frac{x}{4/5} = \frac{5x}{4}$$

$$= \frac{3935}{3945}$$

10. Let x be the required depth of immersion, so that in the upper liquid there is $\frac{h}{n} - x$ and in lower $h - \frac{h}{n} + x$.

Hence we have for the equilibrium

weight of the cylinder$=$weights of the liquids displaced

$$A\,.\,h\,\rho g = A\left(\frac{h}{n} - x\right)\rho_1 g + A\left(h - \frac{h}{n} + x\right)\rho_2$$

where A is the area of cross-section

or

$$h\rho = h\rho_2 + \left(\frac{h}{n} - x\right)(\rho_1 - \rho_2)$$

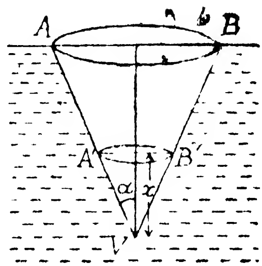
$$\therefore \qquad x = \frac{h}{n} - h\left(\frac{\rho_2 - \rho}{\rho_2 - \rho_1}\right)$$

Clearly $\rho < \rho_2$; for otherwise the whole of the cylinder will sink into lower liquid. Also x must be positive

$$\therefore \quad \frac{\rho_2-\rho_1}{n}-\rho_2+\rho \text{ must be positive}$$

$$\therefore \quad \rho > \rho_2-\frac{\rho_2-\rho_1}{n}.$$

11. Let (V, AB) be the cone with its base in the surface. Let the length of the axis is h and x be the distance of the plane of separation from the vertex.



Now, evidently the C.G. of the cone and these of displaced liquid are in one vertical straight line.

Therefore, for equilibrium,

Weight of the body = sum of the weights of the displaced portions of the liquids

i.e. $\frac{1}{3}\pi h^3 \tan^2 \alpha\, \rho g = \frac{1}{3}\pi x^3 \tan^2 \alpha \,.\, \sigma_1 g + \frac{1}{3}\pi\,(h^3 - x^3)\tan^2 \alpha\, \sigma_2 g$

or $\quad x^3(\sigma_1-\sigma_2) = h^3(\rho-\sigma_2)$

$$x = \left(\frac{\rho-\sigma_2}{\sigma_1-\sigma_2}\right)^{1/3} \text{ of } h.$$

EXAMPLES XII

1. Tension of the string

= Weight of the body − Weight of the displaced liquid

Case I

$$\text{Tension} = 18 - \frac{18}{3} = 12 \text{ lbs. wt.}$$

Case II. $\quad \text{Tension} = 18 - \frac{18 \times 2}{3} = 6 \text{ lbs. wt.}$

2. Tension = Weight of Platinum − weight of displaced liquid

$$= \text{weight of Platinum} - \frac{5}{24} \cdot \frac{\text{wt. of Platinum}}{21}$$

$$-\frac{19}{24}\times\frac{\text{wt. of Platinum}}{21}\times 13$$

$$=\text{wt. of Platinum}\left[1-\frac{5}{24\times 21}-\frac{19\times 13}{24\times 21}\right]$$

$$=\tfrac{1}{2}\text{ weight of Platinum.}$$

3. Let the volume of the silver is V_1 and that of gold is V_2. From the condition of equilibrium

$$V_1\,(10{\cdot}5-{\cdot}85)w=V_2\,(19{\cdot}3-1{\cdot}5)$$

$$V_1\times 9{\cdot}65=V_2\times 17{\cdot}8$$

Therefore the required ratio

$$=\frac{V_1\times 10{\cdot}5}{V_2\times 19{\cdot}3}=\frac{37380}{37249}$$

4. Apparent weight $=1\text{ cwt.}-\frac{1\text{ cwt.}}{7{\cdot}6}$

$$=\frac{66}{76}\text{ cwt.}$$

$$=97\,\frac{5}{19}\text{ lbs. wt.}$$

Suppose x is the required number of lbs. of wood

$$112+x=\frac{112}{7{\cdot}6}+\frac{x}{{\cdot}6}$$

$$x\,.\,\tfrac{2}{3}=112\times\frac{66}{76}$$

$$x=145\,\frac{17}{19}\text{ lbs. wt.}$$

5. Let the specific gravity of the solid is σ.

$$\therefore\quad 1-\frac{1}{\sigma}=\frac{35}{37}$$

$$\frac{1}{\sigma}=1-\frac{35}{37}=\frac{2}{37}$$

$$\sigma=18{\cdot}5.$$

6. Thrust upon the bottom

=weight of the body—weight of the water displaced

$=(30\times1{\cdot}5-30\times1)$

=15 grammes wt.

7. Let V_1 and V_2 are the volumes of gold and silver.

Weight of the water displaced$=(V_1+V_2)\,w$

From the principle of Archimedes

$$\frac{1}{14}\left[V_1\times19{\cdot}25+V_2\times10{\cdot}5\right]w=(V_1+V_2)\,w$$

$$V_1\,(19{\cdot}25-14)=V_2\,(14-10{\cdot}5).$$

$$\frac{V_2}{V_1}=\frac{5{\cdot}25}{3{\cdot}5}=\frac{3}{2}.$$

8. Weight of lead—weight of air displaced by the lead

=weight of wood—weight of air displaced by the wood.

∴ Weight of wood—weight of lead

=weight of air displaced by the wood

—weight of air displaced by the lead

=+ve quantity.

Hence the result.

9. Let V_1, V_2 be the volumes and σ_1, σ_2 the densities of the two bodies A and B respectively.

$$V_1\,\sigma_1=2V_2\,\sigma_2 \ldots\ldots\ldots \text{I}$$

Also $$V_1\,(\sigma_1-1)=V_2\,(\sigma_2-1)$$

or $$\frac{V_1}{V_2}=\frac{\sigma_2-1}{\sigma_1-1}$$

or $$\frac{2\sigma_2}{\sigma_1}=\frac{\sigma_2-1}{\sigma_1-1}$$

∴ $$\frac{\sigma_1}{\sigma_1-1}=\frac{2\sigma_2}{\sigma_2-1}$$

But $$\sigma_1=5/3$$

∴ $$\frac{\frac{5}{3}}{\frac{5}{3}-1}=\frac{2\,\sigma_2}{\sigma_2-1}$$

$$\therefore \quad \frac{2\sigma_2}{\sigma_2-1}=\frac{5}{2}$$

$$\therefore \quad \sigma_2=5.$$

11. **Case I.** There will be no change since the displaced water runs over the side of the vessel.

Case II. Thrust on the base is increased by the weight of the water displaced by the metal.

12. The specific gravity of the wood will be $\frac{2}{3}$. Let V is the required volume of the metal.

From the Principle of Archimedes

$$\left(26\times\frac{2}{3}+V\times8\times\frac{2}{3}\right)W=(26+V)\,W$$

$$\therefore \quad V\left(\frac{16}{3}-1\right)=26-\frac{26\times2}{3}=26\times\frac{1}{3}$$

$$V=2 \text{ cubic inch.}$$

The upward force required = half the weight of the body

$$=\frac{1}{2}\left[26\,.\,\frac{2}{3}+2{\cdot}8\,.\,\frac{2}{3}\right]\times w$$

$$=\frac{42}{3}\times\frac{1000}{1728}\text{ oz. wt.}$$

$$=\frac{875}{1728}\text{ lbs. wt.}$$

13. **Case I.** Thrust is increased by the weight of the displaced water.

Volume of 56 lbs. of iron $=\frac{56}{440}$ cubic feet.

Hence the weight of the displaced water

$$=\frac{56}{440}\times\frac{125}{2}\text{ lbs.}$$

$$=7\,\frac{21}{22}\text{ lbs.}$$

Case II. Thrust is increased by the weight of the iron which is 56 lbs. wt.

14. Let V be the volume of the body and x be the weight/unit volume of water. Then,

$$W' = W - Vx \ldots\ldots\ldots\ldots \text{I}$$

The weight in air (whose sp. gravity is given to be S) will be

$$= W - V \cdot S \cdot x$$
$$= W - S \cdot (W - w').$$

15. Let V be the volume of the body and w the intrinsic weight of water. Then,

$$W' = W - Vw + VSw$$

Since the weight in vacuum is $W + VSw$

$\therefore$ Weight in vacuo $= W + VSw$

$$= V + \frac{S \cdot (W - w')}{1 - S}$$

16. Let W be the weight of the body in air (or vacuo) and w the intrinsic weight of water. Then clearly

$$W_1 = W - VS_1 w \ldots\ldots\ldots\ldots \text{I}$$
$$W_2 = W - VS_2 w \ldots\ldots\ldots\ldots \text{II}$$
$$W_3 = W - VS_3 w \ldots\ldots\ldots\ldots \text{III}$$

Since V is the volume of the body

Subtracting III from I, we get

$$Vw\,(S_3 - S_1) = W_1 - W_3$$

$$\therefore \quad W_2\,(S_3 - S_1) = \frac{W_2\,(W_1 - W_3)}{Vw} \ldots\ldots\ldots \text{IV}$$

Again subtracting I from II, we get

$$Vw\,(S_1 - S_2) = W_2 - W_1$$

$$W_3\,(S_1 - S_2) = \frac{W_3\,(W_2 - W_1)}{Vw} \ldots\ldots \text{V}$$

Similarly, $$W_1\,(S_2 - S_3) = \frac{W_1\,(W_3 - W_2)}{Vw} \ldots\ldots \text{VI}$$

Adding IV, V and VI we get

$$W_1(S_2-S_3)+W_2(S_3-S_1)+W_3(S_1-S_2)$$
$$=\frac{1}{Vw}[W_1(W_3-W_2)+W_2(W_1-W_3)+W_2(W_2-W_1)]=0.$$

17. Let w_1 v and W_1 V denote respectively the weights and volumes of the two given solids. Also suppose that the densities of the three liquids under consideration are ρ_1 ρ_2, ρ_3. Clearly we should have

$$w_1=w-v\rho_1 \ldots\ldots\ldots\ldots \text{I}$$
$$w_2=w-v\rho_2 \ldots\ldots\ldots\ldots \text{II}$$
$$w_3=w-v\rho_3 \ldots\ldots\ldots \text{III}$$
$$W_1=W-V\rho_1 \ldots\ldots\ldots \text{IV}$$
$$W_2=W-V\rho_2 \ldots\ldots\ldots\ldots \text{V}$$
$$W_3=W-V\rho_3 \ldots\ldots\ldots \text{VI}$$

$$\therefore \quad w_1(W_2-W_3)+w_2(W_3-W_1)+w_3(W_1-W_2)$$
$$=V(w-v\rho_1)(\rho_3-\rho_2)+V(w-v\rho_2)(\rho_1-\rho_3)+V(w-v\rho_3)(\rho_2-\rho_1)$$
$$=V[w\{(\rho_3-\rho_2)+(\rho_1-\rho_3)+(\rho_2-\rho_1)\}-V\{\rho_1(\rho_3-\rho_2)+\rho_2(\rho_1-\rho_3)+\rho_3(\rho_2-\rho_1)\}]=0.$$

EXAMPLES XIII

1. Tension of the string = weight of displaced water − weight of the cork

$$=\left(\frac{30}{\cdot 25}-30\right)=(120-30)=90 \text{ grammes wt.}$$

2. Volume of the wood $=\dfrac{6}{\cdot 8\times w}=\dfrac{15}{2w}$ cubic feet.

When the string is just about to break weight of this volume of mixture $=8$ lbs.

Let x be the final proportion of the whole barrel which consists of the fluid of sp. gravity 1·2, we get

$$\frac{15}{2w}\cdot\left\{\frac{(1-x)\times 1+x\times 1\cdot 2}{1-x+x}\right\}\times w=8$$

or $$\frac{15}{2}(1+\cdot 2x)=8$$

or $$\frac{15}{2}+\frac{3}{2}x=8$$

$\therefore$ $$x=\tfrac{1}{3}$$

If $x > \frac{1}{3}$, then the upward thrust of the displaced fluid would be too great and the string would break.

3. Required Force=upward thrust due to the weight of the additional water displaced.

15 lbs.=weight of cylinder of water of length 18 inches.

Therefore required forces$=\frac{6}{18}\times 15=5$ lbs. wt.

4. Total upward thrust on balloon

=weight of air displaced by the coal-gas—weight of the coal-gas

$=4000000\,[1{\cdot}29-{\cdot}52]$

$=3080000$ grammes

$\therefore$ Additional weight$=(3080000-1500000)$ grammes

$=1580000$ grammes wt.

5. Tension of the string

$$=10\times 1{\cdot}25\times\left(1-\frac{1}{14{\cdot}6}\right)\text{ ozs}$$

$$=12{\cdot}5\times\frac{136}{146}\text{ oz}$$

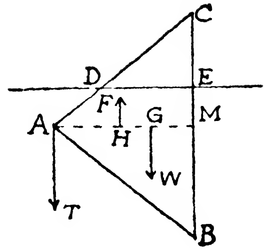
$$=11\frac{47}{73}\text{ ozs.}$$

6. Upward thrust=weight of displaced air—weight of balloon

$$=\left(64000\times\frac{1{\cdot}24}{16}-4480\right)\text{ lbs. wt.}$$

$=480$ lbs. wt.

$\therefore$ Acceleration$=\frac{480\,g}{4480}=\frac{3\,g}{28}$.



7. Take $AB = AC = 2a$ and $\angle CAB = 2\alpha$

Given $CE = \frac{1}{4} CB = \frac{1}{2} CM$.

The forces acting on the triangle in the position of equilibrium are

(i) The tension T,

(ii) The weight W,

(iii) The thrust F.

Let $AM = h$, so that $AG = \frac{2}{3} h$.

Now since $CE = \frac{1}{2} CM$; $DE = \frac{1}{2} AM$.

∴ If the area of triangle DEC be Δ; that of the triangle ABC will be $8\,\Delta$. Also if H be the C.G. of the figure ABEDA, we have

$$MH = \frac{8\,\Delta \cdot \frac{h}{3} - \Delta \cdot \frac{h}{6}}{8\,\Delta - \Delta} = \frac{5}{14} h$$

$$AH = \frac{9}{14} h.$$

Also, $W = 8\,\Delta\ SW$ and $F = 7\,\Delta \cdot w$, w being the intrinsic weight of water. Taking moment about A, we get

$$8\,\Delta\ SW \times \frac{2h}{3} = 7\,\Delta W \times \frac{9h}{14}$$

or $$S = \frac{27}{32}$$

Also
$$\begin{aligned} T &= F - W \\ &= 7\,\Delta\ W - 8\,\Delta\ SW \\ &= 7\,\Delta W - 8\,\Delta \cdot \frac{27}{32} W \\ &= \tfrac{1}{4}\,\Delta\ W \\ &= \frac{1}{27} \cdot 8\,\Delta \cdot \frac{27}{32} W \\ &= \frac{1}{27} W \end{aligned}$$

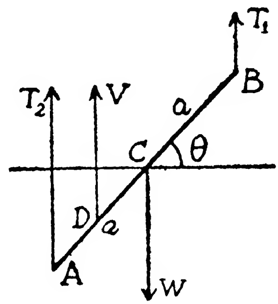
EXAMPLES XIV

1. Let σ be the specific gravity of the rod, and W be its weight, the weight of the displaced fluid is $\frac{2W}{3\sigma}$. Hence, taking moments about the fulcrum, we get

$$W \times 3 = \frac{2}{3} \times \frac{W}{\sigma}(2+2) = \frac{8}{3} \cdot \frac{W}{\sigma}$$

$$\therefore \quad \sigma = \frac{8}{9}$$

2. If the specific weight of water is w.



If $2a$ is the length and K the area of its cross-section,

W (Wt. of the rod) $= 2{\cdot}5 \times 2a.\,Kw$,

V (the upward thrust) $= aK.\,w$

If T_1 and T_2 are the tensions

$$T_1 + T_2 + V = W \ldots\ldots(1)$$

Taking moments about C.

$$T_1 \,.\, a \cos\theta = V \,.\, \frac{a}{2} \cos\theta + T_2 \,.\, \sigma \cos\theta$$

$$i.\,e., \quad T_1 = \frac{V}{2} + T_2 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Putting values of V and W in (1) and (2)

$$T_1 + T_2 + aKw = 5aK \,.\, w \ldots\ldots\ldots\ldots(3)$$

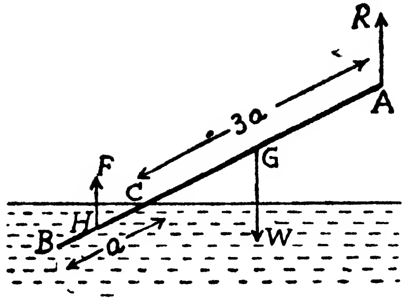
$$\text{and} \quad T_1 = \frac{aKw}{2} + T_2 \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\therefore \quad T_1 + T_2 = 4 \times 2\,(T_1 - T_2)$$

$$\text{or} \quad 7T_1 = 9T_2$$

$$\therefore \quad \frac{T_1}{T_2} = \frac{9}{7}.$$

3. Let AB be the rod of length $6a$ and cross-section α capable of turning about the end A. Let the densities of the rod be ρ, that of the water being unity. The length $AC=4a$ of the rod is out of water, while the length $BC=2a$ is in the water. The resultant thrust of water F acts from the middle point of BC.



Hence taking moments about A in the equilibrium position, we have

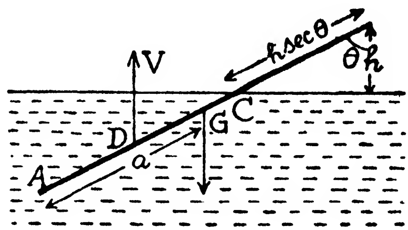
$$W \cdot 3a = F \cdot 5a$$

But $W = 6a\alpha \cdot \rho g$ and $F =$ weight of displaced water $= 2a\alpha \cdot 1g$.

$$\therefore \quad 6a\alpha \cdot \rho g \cdot 3a = 2a\alpha \cdot g \cdot 5a$$

$$\text{whence} \quad \rho = \frac{5}{9}.$$

4. AB is the rod of length $2a$ with its length AC immersed in liquid and B being the fixed point. Let the rod make an angle θ with the vertical. Let the end be at a height h above the free surface.



$$\therefore \quad BC = h \sec\theta.$$

$$\therefore \quad AC = 2a - h\sec\theta.$$

D and G are the mid. points of AC and AB.

$$\therefore \quad AG = a, \ BD = AB - AD = 2a - \tfrac{1}{2} AC.$$

$$= 2a - \frac{2a - h\sec\theta}{2}$$

$$= \frac{2a + h\sec\theta}{2}$$

The forces acting on the rod

(i) The reaction at B.

(ii) The $2a\rho g \cdot \alpha$ through G, where α is the cross-section of the rod.

(iii) The upward thrust of the displaced liquid

$$=(2a-h\sec\theta)\,\sigma g\theta \text{ through D.}$$

Taking moment about A,

$$2a\rho g\,\alpha \cdot \text{BG}\sin\theta=(2a-h\sec\theta)\,\sigma g\alpha \cdot \text{BD}\sin\ \text{C}.$$

or $\quad \sin\theta\,[2a\rho \cdot a-(2a-h\sec\theta)\,\sigma \cdot \frac{1}{2}(2a+h\sec\theta)]\,g\alpha=0$

or $\quad \frac{1}{2}\sin\theta\,[4a^2\rho-(4a^2-h^2\sec^2\theta)\,\sigma]=0$

or $\quad \frac{1}{2}\sin\theta\,[4a^2(\rho-\sigma)+h^2\sigma\sec^2\theta]=0$

either $\sin\theta=0$ *i. e.* $\theta=0$

i. e. rod is vertical

$$4a^2(\rho-\sigma)+h^2\sigma\sec^2\theta=0$$

i. e. $\quad \cos\theta=\dfrac{h}{2a}\sqrt{\dfrac{\sigma}{\sigma-\rho}}$

EXAMPLES XV

1. The meta-centre is at the centre of the ball. If a weight be placed at the highest point, the resulting centre of gravity is now above the centre, *i. e.* above the meta-centre. Hence the equilibrium is now unstable.

2. The meta-centre is at the centre of the common base of the cone and hemisphere. The centre of gravity should not be above the base. Hence, if h be the height and r the radius of the base of the cone, therefore,

$$\tfrac{1}{3}\pi r^2 h\times\frac{h}{4}\leqslant\frac{2}{3}\pi r^3\times\frac{3r}{8}$$

$$\therefore\quad h\leqslant\sqrt{3}r.$$

3. Since the bodies are hollow, we get

$$\tfrac{1}{2}\cdot 2\pi r\cdot l\times\frac{h}{3}\leqslant 2\pi r^2\,\frac{r}{2}$$

i. e. $3\sin^2\alpha\geqslant\cos\alpha.$

This condition is satisfied when $\alpha=45°$

but not when $\alpha=30°$.

4. We know that the meta-centre will be at the centre of the base of the hemisphere. Hence C : G should not be above this centre.

Let h be the height and r the radius of the base,

Case I. $$\pi r^2 h \times \frac{h}{2} \leqslant \frac{2}{3}\pi r^3 \cdot \frac{3r}{8}$$

$$\therefore \quad h^2 \leqslant \frac{r^2}{2}$$

$$h \leqslant \frac{r}{\sqrt{2}}$$

$$i.e. \quad h \leqslant \tfrac{1}{2} r\sqrt{2}.$$

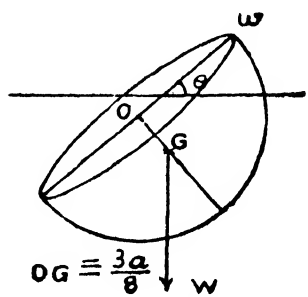
Case II. $$2\pi r h \times \frac{h}{2} \leqslant 2\pi r^2 \times \frac{r}{2}$$

$$h^2 \leqslant r^2,$$

$$h \leqslant r.$$

MISCELLANEOUS EXAMPLES XVI

1. Let the base is inclined at an angle θ with the horizontal. The resultant thrust of water will pass through the centre. Hence the moments of the w and W about the centre must balance.



$$W \cdot \frac{3a}{8} \sin\theta = W \cdot a\cos\theta$$

$$\tan\theta = \frac{8w}{3W}$$

$$\therefore \quad \theta = \tan^{-1}\frac{8w}{3W}.$$

2. Cone is in equilibrium under the forces of buoyancy and its weight only ; they must balance each other. But as the cone can float in every position, the centre of gravity of the hollow cone must coincide with the centre of gravity of the displaced liquid.

Again, the centre of gravity of the surface of the cone lies on the axis at a distance of $\frac{1}{3}$ of the height from the base and the centre of gravity of the base lies at the centre.

Therefore,

$$\frac{h}{4}=\frac{\pi rl\,.\,\frac{h}{3}+\pi r^2\,.\,0}{\pi rl+\pi r^2}$$

$$\text{or}\quad \frac{1}{4}=\frac{\frac{l}{3}}{l+r}$$

$$\text{or}\quad 3l+3r=4l$$

$$l=3r$$

$$\frac{r}{l}=\frac{1}{3}$$

$\therefore$ Sin $\alpha=\frac{1}{3}$, if α is the semi-vertical angle

$\therefore$ Vertical angle$=2\sin^{-1}\frac{1}{3}$.

3. The pressure caused by the bar upon the two parallelopipeds are respectively 75 and 25 lbs. wt.

The total weight to press the larger one down thus

$=175=\frac{7}{3}\times$wt. of water displaced

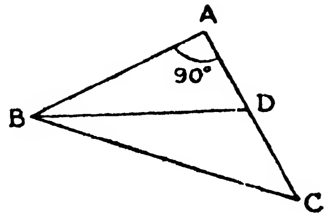
Therefore $\frac{1}{8}$ of it is above the surface.

Similarly the weight to press the smaller one down

$=75$ lbs.$=\frac{3}{4}\times$wt. of water displaced.

Hence $\frac{1}{4}$ of it is above the surface.

4. Let the section of the prism is a right angled triangle ABC, given AB=AC and right angled at A and the point B in the water. Draw BD a horizontal line to meet AC in D.



Since the sp. gr. of the body is $\frac{1}{2}$

$\therefore \quad \triangle ABD = \frac{1}{2} \triangle ABC$

$\therefore \quad AD = \frac{1}{2} AC.$

$\therefore \quad \tan ABD = \dfrac{AD}{AB} = \frac{1}{2}.$

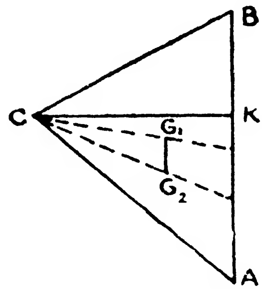
$\therefore \quad \tan CBD = \tan (45 - ABD)$

$$= \frac{\tan 45 - \tan ABD}{1 - \tan 45 \,.\, \tan ABD} = \frac{1-\frac{1}{2}}{1+\frac{1}{2}}$$

$$= \tfrac{1}{3}.$$

Hence the required angle is $\tan^{-1} \frac{1}{3}$.

5. Let the prism float with the edge A immersed. Through C draw CK horizontal to meet AB in K.



Then the weight of the prism will balance with the weight of the liquid displaced. Let G_1, G_2 are the centre of gravity of $\triangle$ ABC and $\triangle$ ACK must be in a vertical line. Therefore AB which is parallel to $G_1 G_2$ must be vertical.

Let σ is the specific gravity of the prism.

Hence,

$$\triangle ABC \times h\sigma w = \triangle ACK \times h \times w$$

or $\quad \sigma = \dfrac{\triangle AKC}{\triangle ABC} = \dfrac{AK \,.\, KC/2}{AB \,.\, KC/2}$

$$= \frac{AK}{AB} = \frac{b \cos A}{c}$$

$$= \frac{\sin B \cos A}{\sin C}.$$

Similarly we can prove with vertex B in water

$$\text{Sp. gr.} = \frac{\sin A \cos B}{\sin C}.$$

6. Suppose V is the vertex and VO is the axis of the cone; the point of the base on the surface cf the water and AOB the diameter of the base through A. Draw VD perp. to the surface of the water and let this surface cut VB in C

$$\therefore \angle AVD = \theta$$

For equilibrium the centre of gravity of the fluid displaced (viz the cone VCA) must be vertically under the surface of the cone VAB.

The horizontal distance from V of the first

$$= \frac{3}{4}\left[\frac{DC+DA}{2}\right] = \frac{3}{8} VA \,.\, \cos\theta \,.\, \frac{\sin(2\theta-60)}{\cos\theta \,.\, \cos(\theta-60)} \ldots \text{I}$$

The horizontal distance of the first we get

$$= \tfrac{2}{3} VO \,.\, \sin(\theta-30)$$

$$= \tfrac{2}{3} VA \,.\, \cos 30 \,.\, \sin(\theta-30) \ldots\ldots\ldots\ldots\ldots\ldots \text{II}$$

Equating I and II, we get

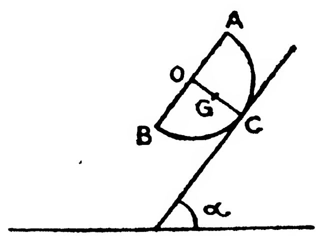
$$\frac{3}{8} \times \frac{2 \sin(\theta-30) \,.\, \cos(\theta-30)}{\cos(\theta-60)} = \sin(\theta-30) \,.\, \frac{\sqrt{3}}{3}$$

$$\therefore \quad 9\left[\cos\theta \,.\, \frac{\sqrt{3}}{2} + \sin\theta \,.\, \tfrac{1}{2}\right] = 4\sqrt{3} \,.\, \cos(\theta-60)$$

$$= 2\sqrt{3}\cos\theta + 6\sin\theta.$$

$$\therefore \quad \tan\theta = \frac{5}{\sqrt{3}}.$$

7. Let AOB be the section of the base of the hemisphere by the plane of the paper, A being the highest point and O the centre, and let G be the C. G. of the bowl. Then, the weight of the bowl will clearly have the greatest effect when OG is



nearly horizontal, i. e. when AB is nearly vertical i. e. when the surface of the water passes through the lowest point B, and then φ is the angle that OB makes with the horizontal or BG with the vertical. Taking moments about the point of contact, C, of the bowl and plane, we then have

Wt. of water $\times$ OC $\sin\alpha$ = wt. of bowl $\times$ (OG $\sin\varphi$ − OC $\sin\alpha$)

Since the weight of the water must pass through O.

$$\therefore \frac{\text{wt. of bowl}}{\text{wt. of water}}=\frac{a\sin\alpha}{\frac{a}{2}\sin\varphi-a\sin\alpha}=\frac{2\sin\alpha}{\sin\varphi-2\sin\alpha}$$

8. Suppose V is the volume of the ball, λV the volume immersed and the weight of a unit volume of water. If T be the tension of the string,

$$T=V\sigma w-\lambda Vw, \text{ where } (O\leqslant\lambda\leqslant 1)$$

also $$T=W+\lambda Vw$$

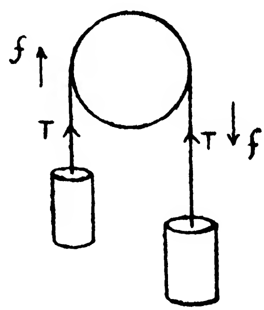
$$\therefore\quad W=V\sigma w-2\lambda Vw$$

$$=V\sigma w\left(1-\frac{2\lambda}{\sigma}\right)$$

$$\therefore\quad V\sigma w=\frac{\sigma W}{\sigma-2\lambda}.$$

The minimum and maximum values are given by putting $\lambda=0$ and $\lambda=1$. Hence the weight lies between W and $\frac{\sigma W}{\sigma-2}$

9. When wood of masses m and m' are tied to the bottoms of buckets as given in the problem. Therefore the equations of motion of the two buckets are given by



$$(M+m)\,f=(M+m)\,g-T\ldots\ldots\text{I}$$

and $$(M+m')\,f=T-(M+m')\,g\ldots\text{II}$$

We have assumed that the bucket with wood of mass m is moving downwards.

Adding I and II

$$f=\frac{(m-m')\,g}{(2M+m+m')}\ldots\ldots\ldots\ldots\ldots\ldots\text{III}$$

Tension in the string attached to the mass m is given by

$$T'=m\left(\frac{1}{\sigma}-1\right)\left(g-f\right).$$

$$=m\left(\frac{1}{\sigma}-1\right)\left[g-\frac{(m-m')\,g}{2M+m+m'}\right]$$

$$=\frac{2m\,(M+m')\,g}{2M+m+m'}\left[\frac{1}{\sigma}-1\right]$$

10. Initially, x being the length immersed, we get

$$x\sigma_1+(h-x)\,\sigma_2=h\rho.$$

Finally x' being the length,

$$x'\sigma_1+(h-x')\,\sigma_3=h\rho$$

$$\therefore\quad x-x'=\frac{h\,.\,(\rho-\sigma_2)}{\sigma_1-\sigma_2}-\frac{h\,(\rho-\sigma_3)}{\sigma_1-\sigma_3}=\frac{h\,(\sigma_1-\rho)(\sigma_3-\sigma_2)}{(\sigma_1-\sigma_2)\,(\sigma_1-\sigma_3)}$$

=cylinder will rise through this distance

11. Let a be the length of the rod and a length x of it be immersed in the liquid.

Then the portion out of liquid$=a-x$.

Let ρ, σ and σ' be the densities of rod, liquid and air respectively.

$\therefore$ For Equilibrium

Weight of the rod=weight of liquid displaced + weight of air displaced

$$a\rho=x\sigma+(a-x)\,\sigma'$$

$$\text{or}\quad x=\frac{a\,(\rho-\sigma')}{\sigma-\sigma'}$$

$$\text{so that } a-x=\frac{\sigma-\rho}{\sigma-\sigma'}\,.\,a\ldots\ldots\ldots\ldots\ldots\ldots1$$

If x the length immersed in the liquid then x changes with the density of atmosphere, so x is a function of σ'.

Ditt. I with respect to σ', we have

$$-\frac{dx}{d\sigma'}=\frac{\sigma-\rho}{(\sigma-\sigma')^2}\cdot a$$

or $$\frac{dx}{d\sigma'}=-\frac{\sigma-\rho}{(\sigma-\sigma')^2}\cdot a=-\frac{(a-x)^2}{a(\sigma-\rho)}$$

If $\rho<\sigma$ otherwise the rod will entirely sink in the liquid of density σ.

$\therefore$ $\frac{dx}{d\sigma'}$ is negative, this shows that with increase in σ', x decreases.

i. e. the rod rises.

Also, $\frac{dx}{d\sigma'}$ varies as $(a-x)^2$

Hence the rate of rising is proportional to the square of the unimmersed length.

12. Let x be the length of the cork which is immersed, so that

$$xw+(h-x)\sigma w=hsw$$

or $$x=\frac{h(s-\sigma)}{1-\sigma}$$

When air is pumped out, let x become y

$$\therefore\ yw=hsw$$

$\therefore$ The cork will sink through a distance

$$=y-x=hs-\frac{h(s-\sigma)}{1-\sigma}.$$

$$=\frac{h\sigma(1-s)}{1-\sigma}.$$

13. Let V is the volume and ρ the density of the body

$$V\rho g=P_1\rho_1 g+(V-P_1)\sigma g\ldots\ldots\text{I}$$

$$V\rho g=P_2\rho_2 g+(V-P_2)\sigma g\ldots\ldots\text{II}$$

$$V\rho g=P_3\rho_3 g+(V-P_3)\sigma g\ldots\ldots\text{III}$$

From II and III, we have

$$(P_3-\rho_2)V\rho=P_2P_3\ (\rho_2-\rho_3)+V\ (P_3-P_2)\ \sigma$$

or $\quad V\ (P_3-P_2)\ (\rho-\sigma)=P_2P_3\ (\rho_2-\rho_3)$

Similarly we have

$$V\ (P_1-P_3)\ (\rho-\sigma)=P_3P_1\ (\rho_3-\rho_1)$$

On adding them we get

$$P_3P_2\ (\rho_2-\rho_3)+P_3P_1\ (\rho_3-\rho_2)+P_1P_2\ (\rho_1-\rho_2)=0$$

$$\frac{\rho_2-\rho_3}{P_1}+\frac{\rho_3-\rho_1}{P_2}+\frac{\rho_1-\rho_2}{P_3}=0$$

14. Let volumes be V_1 and V_2

$\therefore \quad (V_1\sigma_1+V_2\sigma_2)w=a\ldots\ldots\text{I}$

and $\quad [V_1\ (\sigma_1-1)+V_2\ (\sigma_2-1)]\ w=b\ldots\ldots\text{II}$

and $\quad (V_1+V_2)w=a-b\ldots\ldots\text{III}$

From I & III

$$(V_1\sigma_1+V_2\sigma_2)\ (a-b)=(V_1+V_2)\ a$$

or $\quad V_1\ [(a-b)\sigma_1-a]=V_2\ [a-\sigma_2\ (a-b)].$

$\therefore \quad \dfrac{V_1}{V_2}=\dfrac{\sigma_2\ (a-b)-a}{a-\sigma_1\ (a-b)}.$

15. Let W and W′ be the apparent weights, and w the real weight, so that

$$W-\frac{W\sigma}{\rho'}=W-\frac{w\sigma}{\rho}$$

and $\quad W'-\dfrac{W'}{\rho'}\ \sigma=w-\dfrac{w}{\rho'}\ \sigma'.$

$$\frac{W-W'}{\rho'}=\frac{w}{\rho}\left[\frac{\rho-\sigma}{\rho'-\sigma}-\frac{\rho-\sigma'}{\rho'-\sigma'}\right].$$

or $\quad \dfrac{W-W'}{W}=\left[\dfrac{(\sigma-\sigma')\ (\rho-\rho')}{(\rho'-\sigma)(\rho'-\rho')}\right]\left(\dfrac{\rho'-\sigma}{\rho-\sigma}\right)$

$$=\frac{(\sigma-\sigma')\ (\rho-\rho')}{(\rho'-\sigma')\ (\rho-\sigma)}=a \text{ positive quantity}$$

$\therefore \quad W-W'=\dfrac{(\sigma'-\sigma)\ (\rho'-\rho)}{(\rho'-\sigma')\ (\rho-\sigma)}\times \text{former weight.}$

16. Let the sp. gr. of air $\sigma = \cdot 00125$ and suppose the true weight of the water be W and the true weight of the weights is W′. Hence

$$W'\left(1-\frac{\sigma}{\rho}\right)$$

$=$Tension of the string which supports W′

$=$Tension of the string which supports W.

$= W\ (1-\sigma)$

Hence the required connection applied

$$=\frac{W-W'}{W'}=\frac{W}{W'}-1$$

$$=\frac{1-\frac{\sigma}{\rho}}{1-\sigma}-1=\frac{\sigma\left(1-\frac{1}{\rho}\right)}{1-\sigma}$$

$$=\sigma\left(1-\frac{1}{\rho}\right),\ \text{when } \sigma \text{ is very small}$$

$$=\cdot 00125\left(1-\frac{1}{8\cdot 4}\right)=\cdot 000125\times\frac{7.4}{8.4}$$

$=$nearly $\cdot 1$ per cent.

17. Let W be the weight of each ball, and x, y, z their specific gravities.

Hence $W\left(1-\frac{\sigma_1}{x}\right)=W\left(1-\frac{\sigma_1}{y}\right)+W\left(1-\frac{\sigma_1}{z}\right)$

$$\therefore \quad \frac{1}{y}+\frac{1}{z}-\frac{1}{x}=\frac{1}{\sigma_1}\ \ldots\ldots \text{I}$$

$$\frac{1}{z}+\frac{1}{x}-\frac{1}{y}=\frac{1}{\sigma_2}\ \ldots\ldots \text{II}$$

$$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=\frac{1}{\sigma_3}\ \ldots\ldots \text{III}$$

$\therefore$ Adding II & III

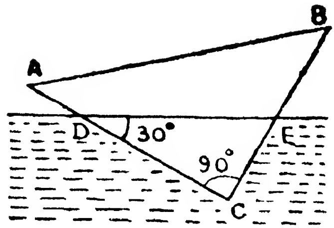
$$\frac{2}{x}=\frac{1}{\sigma_2}+\frac{1}{\sigma_3}$$

so that $x=\dfrac{2\,\sigma_2\sigma_3}{\sigma_2+\sigma_3}$

Similarly $y=\dfrac{2\,\sigma_1\sigma_3}{\sigma_1+\sigma_3}$

and $z=\dfrac{2\,\sigma_1\sigma_2}{\sigma_1+\sigma_2}$

18. Suppose the surface of the liquid cut CA, CB in D and E, so that



$$CE=\tfrac{1}{2}\,CB=\frac{a}{2}$$

and $$CD=CE\tan 60=\sqrt{3}\,.\,\frac{a}{2}\,.$$

$$\therefore\quad \frac{\triangle\,CDE}{\triangle\,CAB}=\frac{CD\,.\,CE}{CA\,.\,CB}$$

$$=\frac{\sqrt{3}\,.\,\frac{a}{2}\,.\,\frac{a}{2}}{a\,.\,b}$$

$$=\frac{\sqrt{3}}{4}\,.\,\frac{a}{b}$$

If the weight of the $\triangle$ CAB be $3w$ it may be replaced by W at each of the angular points A, B, C; also the weight of thd $\triangle$ CDE of liquid will be $\frac{8}{3}\,.\,3W\,.\,\frac{\sqrt{3}}{4}\,.\,\frac{a}{b}$ and it may thus be replaced by $\frac{2\sqrt{3}}{3}\,.\,\frac{a}{b}$ W at each of C, D, E.

Taking moment about C.

$$W\,.\,CA\cos 30-\frac{2\sqrt{3}}{3}\,.\,\frac{a}{b}\,W\,.\,CD\cos 30$$

$$=W\,.\,CB\cos 60-\frac{2\sqrt{3}}{3}\,.\,\frac{a}{b}\,.\,W\,.\,\frac{a}{2}\,.\cos 60$$

or $b\dfrac{\sqrt{3}}{2}-\dfrac{2\sqrt{3}}{3}\dfrac{a}{b}\cdot\dfrac{a}{2}\tan 60\cdot\cos 30=a\cdot\dfrac{1}{2}-\dfrac{2\sqrt{3}}{3}\cdot\dfrac{a}{b}\cdot\dfrac{a}{2}\cdot\dfrac{1}{2}$

$\therefore \qquad 3b\sqrt{3}=2\dfrac{a^2}{b}\sqrt{3}+3a$

i e. $\qquad 2a^2+\sqrt{3}\,ab-3b^2=0$

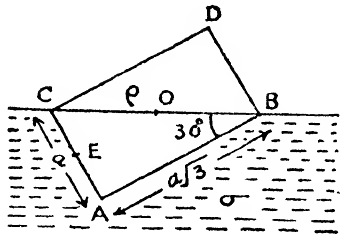
$\therefore \qquad (2a-\sqrt{3}b)(a+\sqrt{3}b)=0$

$\therefore \qquad 2a=\sqrt{3}b$

i e. $\qquad \dfrac{b}{a}=\dfrac{2}{\sqrt{3}}$

$$\frac{CA}{CB}=\frac{2}{\sqrt{3}}.$$

19. Let the Lamina is ABDC. Let E is the fixed point, the middle point of the shorter side AC. Let CB be the diagonal in the surface of the liquid and A the vertex under liquid. Let AC $=a$



and $\qquad AB=a\sqrt{3} \qquad \therefore \quad \angle ABC=30.$

Suppose ρ be sp. gr. of Lamina and σ that of liquid.

The weight of Lamina $=a^2\sqrt{3}\rho$. This wt. acts through θ, the middle point of BC.

The weight of the displaced liquid $=\frac{1}{2}a^2\sqrt{3}\sigma$, and this may be replaced by $\dfrac{a^2\sqrt{3}\sigma}{6}$ acting upwards at A, B & C.

Taking moment about E, we get

$$a^2\sqrt{3}\rho\times\frac{\sqrt{3}a}{2}\cdot\cos 30=$$

$$\frac{a^2\sqrt{3}\sigma}{6}\left[\frac{a}{2}\cos 60+\left(\frac{a}{2}\cos 60+\sqrt{3}\,a\cos 30\right)-\frac{a}{2}\cos 60\right]$$

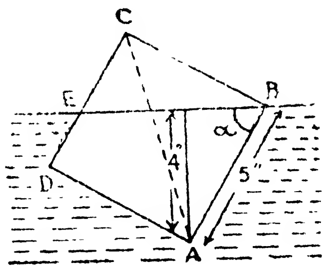
$$=\frac{a^3\sqrt{3}}{6}\sigma\left[\tfrac{1}{2}\cos 60+\sqrt{3}\cos 30\right]$$

or $$\frac{3a^3\rho}{2} \cdot \frac{\sqrt{3}}{2} = \frac{a^3\sqrt{3}\sigma}{6} \times \frac{7}{4}$$

$\therefore$ $$\frac{\rho}{\sigma} = \frac{7}{18}.$$

Hence the downward thrust on E

$$= \frac{a^2\sqrt{3}\sigma}{2} - a^2\sqrt{3}\,\rho$$
$$= \tfrac{9}{7}\, a^2\sqrt{3}\rho - a^2\sqrt{3}\rho$$
$$= \tfrac{2}{7}\, a^2\sqrt{3}\rho$$
$$= \tfrac{2}{7} \times \text{weight of Lamina.}$$



20. Let ABCD is a square of side 5 inches. Let the surface of water through B cut CD in E, and let $\angle$ ABE $= \alpha$.

So that $\sin\alpha = \frac{4}{5}$; $\cos\alpha = \frac{3}{5}$

CE $=$ CB . $\cot\alpha = 5 \cdot \frac{3}{4} = \frac{15}{4}$

Area of $\triangle$ ECB $= \frac{1}{2}$. EC . CB
$$= \tfrac{1}{2} \cdot \tfrac{15}{4} \cdot 5 = \tfrac{75}{8}$$

Now the upward thrust of the water is $w \times$ area of DEBA, and is equal to $w \times$ area of ABCD acting upward at O the middle point of AC, and $w \times$ area of ECB acting downwards at its C G. This latter can be replaced by $\frac{25}{8}\, w$ at each of the vertices E, B, C

Also the weight of the square is $25\,\rho w$ at O. Hence taking moment about A, we get

$$25\,(\rho - 1)\, w \times \frac{AC}{2} \cdot \cos(\alpha + 45) + \frac{25}{8}\, w$$
$$[AB\cos\alpha + (AB\cos\alpha - BC\sin\alpha)$$
$$-(AD\sin\alpha - DE\cos\alpha)] = 0$$

or $$8\,(1-\rho)\,\frac{5\sqrt{2}}{2}\left[\frac{\cos\alpha - \sin\alpha}{\sqrt{2}}\right] = 10\cos\alpha - 10\sin\alpha + \frac{5}{4}\cos\alpha$$

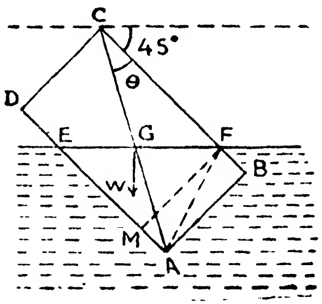
or $4(1-\rho)(1-\tan\alpha)=2-2\tan\alpha+\frac{1}{4}$

$$=\frac{9-8\tan\alpha}{4}$$

$$\therefore \quad (1-\rho)=\frac{1}{16}\cdot\left[\frac{8\tan\alpha-9}{\tan\alpha-1}\right]=\frac{1}{16}\cdot\left[\frac{32-27}{4-3}\right]$$

$$\therefore \quad \rho=\frac{11}{16}.$$

21. Let ABCD be the rectangle movable about C. EF is the horizontal line in which the surface of the liquid cuts the rectangle. $BC=b$, $CD=a$. Let $BFE=x$, $\angle ACB=\theta$.



Draw FM perp. to AD. Since EM, MF are equally inclined to the horizon, $EM=MF=a$.

Since half the rectangle is in liquid, the surface EF of the liquid must pass through G, the C. G. of the rectangle.

$$\therefore \quad \tfrac{1}{2}ab=\text{rect. AMFB}+\triangle\text{EFM}$$
$$=x\,a+\tfrac{1}{2}a^2$$
$$\therefore \quad x=\tfrac{1}{2}(b-a),\ \text{and } AE=\tfrac{1}{2}(a+b)$$

Let ρ and σ be the densities of the rect. and liquid respectively.

The weight W of the rectangle$=abg\rho$ and acts vertically downward through G.

The weight W′ of the liquid $ABFM=ax\,.\,g\sigma=\frac{1}{2}a(b-a)g\sigma$ and acts vertically upwards through the mid-point of AF.

The weight W″ of the liquid $EFM=\frac{1}{2}EM.\,FM\,g\sigma=\frac{1}{2}a^2g\sigma$.

Also W″ acting at the C. G. of the $\triangle$ EFM is equivalent to $\frac{1}{3}$ W″ at each of the angular points E, F, M.

Taking moment about C.

$$\frac{1}{2}W\,AC\cos(\theta+45)=\frac{W''}{3}\{(b-x)\cos 45+AC\cos(\theta+45)$$

$$-a\cos 45^\circ+(b-a-x)\cos 45^\circ\}$$
$$+\frac{W'}{2}\{AC\cos(\theta+45)+(b-x)\cos 45^\circ\}$$

or $$\frac{W}{2\sqrt{2}}(b-a)=\frac{W''}{3\sqrt{2}}(b-x+b-a-x-a+b-a-x)$$
$$+\frac{W'}{2\sqrt{2}}(2b-a-x)$$

or $$\frac{W}{2}(b-a)=W''(b-a-x)+\frac{W'}{2\sqrt{2}}(2b-a-x)$$

or $$\frac{W}{2}(b-a)=\frac{W''}{2}(b-a)+\frac{W'}{4}(3b-a)$$

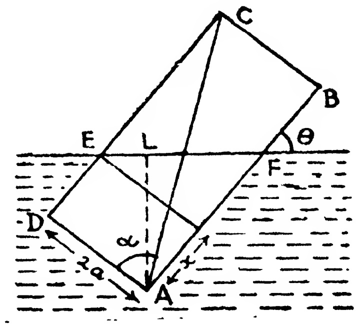
or $$\frac{ab}{2}(b-a)\rho g=\frac{1}{4}a^2\sigma(b-a)g+\frac{1}{8}a(b-a)(3b-a)\sigma g$$

or $$b\rho=\tfrac{1}{2}a\sigma+\tfrac{1}{4}(3b-a)\sigma$$

$$\therefore\quad \frac{\rho}{\sigma}=\frac{3b+a}{4b}.$$

22. $AB=2b$, $AD=2a$

Let $\angle DAC=\alpha$



$$\cos\alpha=\frac{a}{\sqrt{a^2+b^2}},\ \sin\alpha=\frac{b}{\sqrt{a^2+b^2}}$$

Let the water surface cut DC in E and AB in F; draw EK parallel to AD.

Also let $AK=DE=x$

$AL=C$ (given)

Therefore from $\triangle$ ALF

$$\sin\theta=\frac{AL}{AF}=\frac{C}{AF}$$

$$\therefore\quad \frac{C}{\sin\theta}=AF=AK+KF=x+2a\cot\theta\ldots\ldots\ldots \text{I}$$

Also, Area of $\triangle KEF=\tfrac{1}{2}\,.\,2a\cot\theta\,.\,2a=2a^2\cot\theta$

The forces acting are

$4ab\, w\sigma$ at O, the middle point of AC downwards,

$x\,.\,2aw$ at the middle point of AE and upwards,

$\frac{2a^2}{3}\,.\cot\theta\,.\,w$ upwards at each of the points K, E, F.

Hence, by taking moment about A, we have

$$4\,ab\sigma\,.\,\frac{AC}{2}\,.\sin(\theta-\alpha)$$

$$=2ax\,.\,\frac{2a\sin\theta-x\cos\theta}{2}+\frac{2a^2}{3}\,.\cot\theta\left[(2a\sin\theta-x\cos\theta)\right.$$

$$\left.-x\cos\theta-\frac{c}{\sin\theta}\,.\cos\theta\right]$$

$$\therefore\quad 4\,b\sigma\,[a\sin\theta-b\cos\theta]=x\,(2a\sin\theta-x\cos\theta)$$

$$+\frac{2a}{3}\cot\theta\,[2a\sin\theta-2x\cos\theta-c\cot\theta]$$

$$=\left(\frac{c}{\sin\theta}-2a\cot\theta\right)\left[\frac{2a}{\sin\theta}-c\cot\theta\right]$$

$$+\frac{2a}{3}\,.\cot\theta\,[2a\sin\theta-3c\cot\theta+4a\cot\theta\,.\cos\theta\,]$$

Substituting for x

$$\therefore\quad 12\,b\sigma\sin^2\theta\,(a\sin\theta-b\cos\theta)$$

$$=3\,[c-2a\cos\theta]\,[2a-c\cos\theta]+2a\cos\theta\,[2a\sin^2\theta$$

$$-\,3c\cos\theta+4a\cos^2\theta]$$

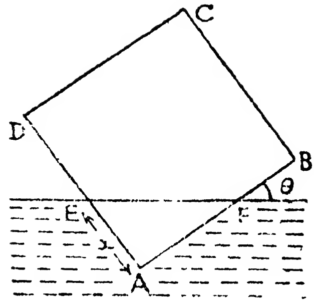
$$=6\,ac-3c^2\cos\theta-12a^2\cos\theta+6\,ac\cos^2\theta+4a^2\cos\theta\,.$$

$$\sin^2\theta-6\,ac\cos^2\theta+8a^2\cos^3\theta$$

$$=6\,ac-3c^2\cos\theta-4a^2\cos\theta\,(3-\sin^2\theta-2\cos^2\theta)$$

$$=6\,ac-3c^2\cos\theta-4a^2\cos\theta\,(2-\cos^2\theta).$$

23. Let the angular point, A of the square ABCD be under the water, and let AB be inclined at θ to the horizontal.



Let the water line meet AD, AB in E and F and let $AE = x$ and hence $AF = x \cot\theta$

From the condition of floating bodies,

$$\therefore \quad \tfrac{1}{2} x^2 \cot\theta = a^2\rho \ldots\ldots \mathrm{I}$$

a is the side of the square.

If the weight of the rectangle be 3W, then the upward thrust of the liquid is given by W at each of A, E, and F.

Taking moment about A, we get

$$3W \frac{\sqrt{2}a}{2} \cdot \cos(\theta+45) = W\,[x \cot\theta \cdot \cos\theta - x\sin\theta]$$

$$\therefore \quad \frac{3a}{2}(\cos\theta - \sin\theta) = x \cdot \left(\frac{\cos^2\theta - \sin^2\theta}{\sin\theta}\right)$$

or $\quad 3a\sin\theta = 2x(\cos\theta + \sin\theta)$

or $\quad \cos\theta - \sin\theta = 0$

i e. $\quad \theta = 45$

$$9a^2\sin^2\theta = 4x^2(\cos\theta + \sin\theta)^2$$

From I

$$9a^2\sin^2\theta = \frac{8a^2\rho}{\sigma} \cdot \tan\theta\,(\cos\theta + \sin\theta)^2$$

$$\therefore \quad 9\sigma\sin\theta \cdot \cos\theta = 8\rho\,(1 + 2\sin\theta \cdot \cos\theta)$$

$$9\sigma\sin 2\theta = 16\rho \cdot (1 + \sin 2\theta).$$

$$\sin 2\theta = \frac{16\rho}{9\sigma - 16\rho}$$

provided $\quad 16\rho < 9\sigma - 16\rho \quad$ *i.e.* $\quad 32\rho < 9\sigma$

This gives two values of 2θ each less than 180 and hence two values of θ, less than 90.

One value of θ has been found to be 45. Hence there are three positions of equilibrium.

25. Let W be the counterpoise, A the area of the cross-section of the cylinder and l and σ its length and density, K the total length of the chain and w_1 its weight per unit length, and yet y be the length of the chain between the pulley and cylinder when a length x of the cylinder is in the water.

Then $$W+(K-y)\,w_1=Al\sigma w-Axw+yw_1\ldots\ldots\text{I}$$

Also, if h be the height of the pulley above the water, then

$$h=y+l-x\ldots\ldots\text{II}$$

Substituting for y in I, we have

$$W=Al\sigma w-Axw+w_1\,(2h+2x-2l-K)$$

This equation is true for all values of x *i.e.* the cylinder will rest with any length immersed if

$$W=Al\sigma w+w_1\,(2h-2l-K)$$

$$O=-Aw+2w_1$$

$$\therefore\quad A=\frac{2w_1}{w}$$

$$W=w_1\,[2h-K-2l\,(1-\sigma)]$$

26. Let O be the centre of fixed sphere, O′ that of the movable hemisphere, and A the point of contact. Let $2r$ and r be the radii of the spheres.

The upper body of solid would be in stable equilibrium if the height of its centre of gravity above A were such that

$$\frac{1}{h}>\frac{1}{r}+\frac{1}{2r}\quad i.e. > \frac{3}{2r}$$

$$i.e.\ h<\frac{2r}{3}.$$

Now when the upper hemisphere is turned slightly, the water thrust on it still passes through the centre O′. Hence, as far as the question of stability is concerned, we may replace the water by an equal weight at O′.

Thus, if W′ be the weight of the water and W that of the hemisphere, we have, since the C.G. of the latter bisects AO′.

$$\frac{W'\times r+W\cdot\frac{r}{2}}{W'+W}<\frac{2r}{3}\quad i.e.\ 3W'+\frac{3W}{2}<2W'+2W.$$

Hence $$W'<\frac{W}{2}.$$

EXAMPLES XVII

1. Weight of the water filling the bottle $=187{\cdot}63-7{\cdot}95$
$=179{\cdot}68$ grs.

Weight of the liquid filling the bottle $=142{\cdot}71-7{\cdot}95$
$=134{\cdot}76$ grs.

Hence the specific gravity of the given liquid

$$=\frac{134{\cdot}76}{179{\cdot}68}={\cdot}75.$$

2. Weight of the water $=983$ grains

Weight of the Alcohol $=773$ grains

Hence the sp. gr. of the Alcohol $=\frac{773}{983}={\cdot}7864.$

3. Weight of the solid (iron) $=10$ gms.

Weight of the bottle when full of water $=44$ gms.

Weight of the bottle when it contains the solid and is filled up with water $=52{\cdot}7$ gms.

Hence

$52{\cdot}7-44=$ wt. of the solid $-$ wt. of water displaced by the solid.

$\therefore$ Weight of the water displaced by the solid

$$=10+44-52{\cdot}7=1{\cdot}3 \text{ gm.}$$

Hence the sp. gravity of iron

$$=\frac{\text{Weight of the iron}}{\text{Wt. of displaced liquid}}$$

$$=\frac{10}{1{\cdot}3}=7{\cdot}6923.$$

4. Weight of the water displaced by the solid

$$=38{\cdot}4+22{\cdot}3-49{\cdot}8$$
$$=10{\cdot}9 \text{ grms.}$$

Hence the sp. gr. of the solid

$$=\frac{22{\cdot}3}{10{\cdot}9}$$
$$=2{\cdot}0458.$$

5. Weight of the water displaced by the solid

$$=212+50-254$$
$$=8 \text{ grains}$$

Hence the sp. gr. of the metal $=\frac{50}{8}=6{\cdot}25$.

6. Let the weight of the bottle be W.

When filled with water, let the weight be W′.

When filled with the given liquid, let its weight be W″.

Let σ_1 be the real sp. gr. of the liquid, and θ that of the substance of which the weights are made. Let α be the sp. gr. of air.

Wt. in air of W′ = wt. in air of bottle + wt. in air of water.

$$W'\left(1-\frac{\alpha}{D}\right)=\text{wt. in air of bottle}+\text{wt. of water }(1-\alpha)$$

$$W''\left(1-\frac{\alpha}{D}\right)=\text{wt. in air of bottle}+\text{wt. of liquid}\left(1-\frac{\alpha}{\sigma_1}\right)$$

$$W\left(1-\frac{\alpha}{D}\right)=\text{wt. in air of bottle.}$$

$$\sigma=\frac{W''-W}{W'-W}=\frac{\text{wt. of liquid}\,.\left(1-\frac{\alpha}{\sigma_1}\right)}{\text{wt. of water }(1-\alpha)}=\sigma_1\times\frac{1-\frac{\alpha}{\sigma_1}}{1-\alpha}.$$

$$\therefore \qquad \sigma=\frac{\sigma_1-\alpha}{1-\alpha}$$

$$\therefore \quad \sigma_1=\sigma-\alpha\,(\sigma-1).$$

EXAMPLES XVIII

1. Weight of the water displaced by the body $=732-252$
$=480$ grammes

From the principle of Archimedes

$$\text{Sp. gr. of the solid}=\frac{\text{weight of the body}}{\text{weight of the displaced water}}$$

$$=\frac{732}{480}=1\cdot525.$$

2. Weight of the displaced water $=2\cdot4-1\cdot6=\cdot8$

Hence the sp. gr. $=\dfrac{2\cdot4}{\cdot8}=3.$

3. Weight of the displaced turpentine $=3-1\ 86=1\cdot14$

Specific gravity of the cupric sulphate

$$=\frac{\text{wt. of cupric sulphate}}{\text{wt. of displaced water}}$$

$$=\frac{\text{wt. of cupric sulphate}}{\text{wt. of turp. displaced}}\times\left(\frac{\text{wt. of turp. displaced}}{\text{wt. of water displaced}}\right)$$

$$=\frac{3}{1\cdot14}\times\text{sp. gr. of turp.}$$

$$=\frac{3}{1\cdot14}\times\cdot88$$

$$=2\tfrac{6}{19}.$$

4. Weight of the displaced Naphtha $=432\cdot5-9$
$=423\cdot5$ grm.

$$\text{Sp. gr. of potassium}=\frac{\text{wt. of potassium}}{\text{wt. of displaced Naptha}}\times\text{sp. gr. of Naptha.}$$

$$=\frac{432\cdot5}{423\cdot5}\times\cdot847$$

5. Let w be the weight of lead in air.

Then the weight of water displaced by the lead and wood $=w+120-$combined wt. in water

$$=w+120-20=w+100.$$

Hence the weight of water displaced by the lead $=w-30$.
Hence the weight of water displaced by the wood

$$=(w+100)-(w-30)=130.$$

$\therefore$ Sp. gravity of wood $=\frac{120}{130}=\frac{12}{13}$.

6. Weight of solid in water $=6-8=-2$ lb.
Hence weight of water displaced by the solid

$$=4-(-2)=6 \text{ lb}.$$

Hence

$$\text{sp. gr.}=\tfrac{4}{6}=\tfrac{2}{3}$$

7. Let σ be the sp. gr. of the attached substance
Therefore

$$200=200\left(1-\frac{1}{\sigma}\right)+300\left(1-\tfrac{1}{5}\right)$$

$$200=200-\frac{200}{\sigma}+300-60$$

$$\frac{200}{\sigma}=240$$

$$\sigma=\frac{200}{240}=\frac{5}{6}$$

8. Let σ and σ' be the specific gravity of glass and alcohol.
Therefore,

$$22=47\left(1-\frac{1}{\sigma}\right)\text{......I}$$

or

$$\frac{22}{47}=1-\frac{1}{\sigma}$$

or

$$\frac{1}{\sigma}=1-\frac{22}{47}=\frac{25}{47}$$

Also,

$$25{\cdot}8=47\left(1-\frac{\sigma'}{\sigma}\right)\text{......II}$$

$$\frac{25{\cdot}8}{47}=1-\frac{\sigma'}{\sigma}$$

or
$$\frac{\sigma'}{\sigma}=1-\frac{25\cdot8}{47}=\frac{21\cdot2}{47}$$
$$\sigma'=\frac{21\cdot2}{47}\times\frac{47}{25}$$
$$=\cdot848$$

9. Let σ be the specific gravity of the olive oil. Therefore
$$1=1\cdot09\left(1-\frac{\sigma}{11\cdot4}\right)$$
$$1=1\cdot09-\sigma\,.\,\frac{(1\cdot09)}{11\cdot4}$$
$$\sigma\,.\,\frac{(1\cdot09)}{11\cdot4}=\cdot09$$
$$\sigma=\frac{\cdot09}{1\cdot09}\times11\cdot4$$
$$\sigma=\cdot9413.$$

10. Weight of water displaced by glass
$$=665\ 8-465\cdot8$$
$$=200\text{ gm.}$$

Weight of sulphuric acid displaced by glass
$$=665\cdot8-297\ 6$$
$$=368\cdot2$$

$\therefore$ Sp. gr. of sulphuric acid $=\dfrac{368\cdot2}{200}$
$$=1\cdot841.$$

11. Weight of water displaced by sugar and wax
$$=40+5\cdot76-14\cdot76=31.$$

Therefore, weight of water displaced by wax $=\dfrac{5\cdot76}{\cdot96}=6$

Weight of water displaced by sugar
$$=31-6=25$$

$\therefore$ Sp. gr. of the sugar $=\frac{40}{25}=1\cdot6.$

12. Weight of water displaced by copper and wax

$$=72+18-62=28$$

Weight of water displaced by wax$=\frac{18}{\cdot 9}=20$

Weight of water displaced by copper

$$=28-20=8$$

Sp. gravity of copper$=\frac{72}{8}=9.$

13. Let V be the volume of the marble and σ the specific gravity of the oil. Then we know that

$$V\,(2{\cdot}84)-V=92$$

$$V\,(2{\cdot}84-1)=92 \ldots\ldots\ldots\ldots \mathrm{I}$$

$$V\,(2{\cdot}84)-V\sigma=98{\cdot}5$$

$$V\,(2{\cdot}84-\sigma)=98{\cdot}5 \ldots\ldots \mathrm{II}$$

Dividing I by II

$$\frac{2{\cdot}84-1}{2{\cdot}84-\sigma}=\frac{92}{98{\cdot}5}$$

or $$\frac{1{\cdot}84}{2{\cdot}84-\sigma}=92$$

or $$1{\cdot}84=92\,(2{\cdot}84-\sigma)$$

$$92\,\sigma=92\,(2{\cdot}84)-1{\cdot}84$$

$$\sigma={\cdot}87 \quad \therefore \quad V=\frac{92}{1{\cdot}84}=50.$$

14. Let V is the volume and σ its specific gravity

$$V\sigma-{\cdot}8V=18$$

$$V\,(\sigma-{\cdot}8)=18 \ldots\ldots\ldots\ldots \mathrm{I}$$

Also, $V\sigma-1{\cdot}2V=12$

$$V\,(\sigma-1{\cdot}2)=12 \ldots\ldots\ldots\ldots \mathrm{II}$$

Divide I by II

$$\frac{\sigma-{\cdot}8}{\sigma-1{\cdot}2}=\frac{18}{12}=\frac{3}{2}$$

$$2\sigma-1{\cdot}6=3\sigma-3{\cdot}6$$

From I $\sigma=2$

$$\therefore \quad V\ (2-\cdot 8)=18$$

$$\therefore \quad V=\frac{18}{1\cdot 2}=15$$

True weight $=15\times 2=30$.

15. If W and W′ be the true weights of the sinker and substances and σ and σ' their specific gravities. With the usual formula,

$$W\left(1-\frac{1}{\sigma}\right)+W'\left(1-\frac{1}{\sigma'}\right)=4W' \ldots\ldots\ldots\ldots \mathrm{I}$$

and $$W\left(1-\frac{1}{\sigma}\right)=5W' \ldots\ldots\ldots \mathrm{II}$$

Subtracting II from I

$$W'\left(1-\frac{1}{\sigma'}\right)=-W'$$

$$1-\frac{1}{\sigma'}=-1$$

$$\frac{1}{\sigma'}=2 \qquad \sigma'=\tfrac{1}{2},$$

16. Let σ and σ' be the specific gravities of the body and the liquid, then

$$4W\left(1-\frac{1}{\sigma}\right)=W \ldots\ldots\ldots\ldots \mathrm{I}$$

or $$4-\frac{4}{\sigma}=1$$

$$\frac{4}{\sigma}=3$$

$$\therefore \qquad \sigma=\tfrac{4}{3}.$$

Also $$W\left(1-\frac{1}{\sigma}\right)=\frac{4}{3}\,W\left(1-\frac{\sigma'}{\sigma}\right) \ldots\ldots\ldots \mathrm{I}$$

$$1-\frac{3}{4}=\frac{4}{3}\left(1-\sigma'\cdot\frac{3}{4}\right)$$

$$1-\tfrac{3}{4}=\tfrac{4}{3}-\sigma'$$

$$\sigma'=\tfrac{4}{3}-\tfrac{1}{4}=\tfrac{13}{12}.$$

Hence $\sigma=\frac{4}{3}$ and $\sigma'=\frac{13}{12}$

17. Weight of water displaced by the crown $=\dfrac{7\frac{1}{2}}{19{\cdot}2}$

Hence, weight in water $=7\frac{1}{2}-\dfrac{7\frac{1}{2}}{19{\cdot}2}=\dfrac{15}{2}\left(1-\dfrac{1}{19{\cdot}2}\right)$

$$=\frac{15}{2}\times\frac{182}{192}$$

$$=7\tfrac{7}{64}\text{ lbs.}$$

Let the crown contained x lbs. of gold and y lbs. of silver.

$$\therefore\quad x+y=7\tfrac{1}{2}\text{............I}$$

Also $\dfrac{x}{19{\cdot}2}+\dfrac{y}{10{\cdot}5}=$ wt. of displaced water

i. e. $\dfrac{x}{19{\cdot}2}+\dfrac{y}{10{\cdot}5}=7\dfrac{1}{2}-7\dfrac{1}{34}=\dfrac{8}{17}$ II

Solving I and II we get

$$x=\frac{96}{17}\text{ and }y=\frac{63}{34}.$$

18. Let W be the weight of the body and w being the weight of the sinker and body together when placed in water. Let σ_1 be the true sp. gr. and D that of the "Weights."

Hence,

$$1-w\left(1-\frac{\alpha}{D}\right)=\text{wt. of body}-\text{wt. of displaced water.}$$

Also $\quad w\left(1-\dfrac{\alpha}{D}\right)=\text{wt. of body}\left(1-\dfrac{\alpha}{\sigma_1}\right)$

$$\therefore\quad \frac{w}{W}=\frac{\text{wt. of body}-\text{wt. of displaced water}}{\text{wt. of body}\left(1-\dfrac{\alpha}{\sigma_1}\right)}$$

$$=\frac{1}{1-\dfrac{\alpha}{\sigma_1}}-\frac{1}{1-\dfrac{\alpha}{\sigma_1}}\times\frac{\text{wt. of displaced water}}{\text{wt. of the body}}\text{.........I.}$$

Also $$\frac{W}{W-w}=\sigma$$

$$\therefore \quad \frac{1}{\sigma}=1-\frac{w}{W}$$

From I

$$1-\frac{1}{\sigma}=\frac{1}{1-\frac{\alpha}{\sigma_1}}\left(1-\frac{1}{\sigma_1}\right)=\frac{\sigma_1-1}{\sigma_1-\alpha}$$

$$\therefore \quad \sigma_1=\sigma-\alpha\ (\sigma-1).$$

Hence this result is greater by an amount $\alpha\ (\sigma-1)$

EXAMPLES XIX

1. Let the required volumes be V_1, V_2, V_3 below the graduation 1, 1·1 and 1·2. Also, let w is weight of per unit volume.

Therefore,

$$V_1\times w=\tfrac{2}{16}\ldots\ldots\ldots\ldots\text{I}$$

but $w=\frac{125}{8}$

$$\therefore \quad V_1=\frac{2}{16}\times\frac{2}{125}=\frac{2}{1000}\text{ cub. ft.}$$

$$V_1=\frac{2\times 12\times 12\times 12}{1000}\text{ cubic inches}$$

$$=3{\cdot}456\text{ cubic inches.}$$

Similarly $V_2\times 1{\cdot}1w=\tfrac{2}{16}\ldots\ldots\ldots\ldots\text{II}$

$$V_2=\frac{2}{16}\times\frac{10}{11}\times\frac{2}{125}=\frac{2}{1100}\text{ cub. ft.}$$

$$=3{\cdot}1418\text{ cub. inches.}$$

Also $V_3\times 1{\cdot}2w=\tfrac{2}{16}\ldots\ldots\ldots\ldots\text{III}$

$$V_3=\frac{2}{16}\times\frac{10}{12}\times\frac{2}{125}=\frac{1}{600}\text{ cub. ft.}$$

$$=\frac{12\times 12\times 12}{600}\text{ cub, inches.}$$

$$=2{\cdot}88\text{ cub. inch.}$$

2. Let w is the weight of the common hydrometer and V its volume. Also, let σ is the sp. gr. of milk. Therefore,

$$W=\frac{9}{10}V\,.\,I\,.\,w-\frac{90}{103}V\,.\,\sigma\,w\ldots\ldots\ldots\ldots I$$

$$\therefore\quad \sigma=\frac{9}{10}\times\frac{103}{90}=\frac{103}{100}=1{\cdot}03$$

3. Let W is the weight of the hydrometer and V its volume. Let A is the area of cross-section. Suppose it exposed x inches in a liquid of density $1{\cdot}3$. Therefore,

$$W=(V-4A)\times 1{\cdot}2W=(V-8A)\times 1{\cdot}4W=(V-xA)\times 1{\cdot}3W\ldots I$$

$$\therefore\quad 1{\cdot}2V-4{\cdot}8A=1{\cdot}4V-11{\cdot}2A$$

$${\cdot}2V=11{\cdot}2V-4{\cdot}8A$$

$$2V=64A\ldots\ldots\ldots\ldots II$$

Also $$1{\cdot}2V-4{\cdot}8A=1{\cdot}3V-1{\cdot}3\,xA$$

$$V=13\,xA-48A=(13x-48)\,A$$

$$V=(13x-48)\,.\,\frac{2V}{64}$$

$$\therefore\quad 2\,(13x-48)=64$$

$$13x-48=32$$

$$13x=32+48=80$$

$$x=\frac{80}{13}=6\,\frac{2}{13}\text{ inches.}$$

4. Let the weight of the hydrometer is W and V its volume.

$$W=V\times 1{\cdot}6W=(V+A)\times 1{\cdot}3W=(V+2A)\,\sigma W\ldots\ldots\ldots I$$

where σ is the sp. gr. corresponding to the highest mark.

$$1{\cdot}6V=1{\cdot}3V+1{\cdot}3\,A$$

$$3V=13\,A\ldots\ldots\ldots\ldots II$$

Also $$1{\cdot}6V=\sigma V+2\sigma\,A$$

$$(16-10\sigma)\,V=20\sigma\,A.$$

or $$(16-10\sigma)\,V=20\sigma\,.\,\frac{3V}{13}$$

or $$13\,(16-10\sigma)=60\sigma$$
$$190\,\sigma=208$$
$$\therefore\quad \sigma=\frac{208}{190}=1{\cdot}0947$$

5. Let x cubic cm. be the unimmersed volume. Therefore,
$$(12-x)\,({\cdot}85)=9$$
$$85\,x=1020-900=120$$
$$x=\frac{120}{85}=\frac{24}{17}=1\,\frac{7}{17}\text{ cub. cm.}$$

6. If W be the original weight, and xW the weight lost. Hence the volume lost is $\frac{xW}{\sigma W}$ where σ is the sp. gr. of the substance of the bulb. If V be the volume below the point which is now in the surface of the water, we know that
$$W=V\times 1{\cdot}002\,w\ldots\ldots\ldots\ldots\text{I}$$
Also $$W-xW=\left(V-\frac{xW}{\sigma w}\right).\,w\ldots\ldots\ldots\text{II}$$
$$W\,(1-x)=\left(V-\frac{xW}{\sigma w}\right).\,w$$
$$V\,.\,(1{\cdot}002)\,w\,(1-x)=\left[V-\frac{x\,.\,V\,.\,(1{\cdot}002)\,w}{\sigma w}\right].\,w\,.$$
$$1{\cdot}002\,(1-x)=\left[1-\frac{1{\cdot}002\,x}{\sigma}\right]$$
$$1{\cdot}002\,\sigma-1{\cdot}002\,.\,\sigma x=\sigma-1{\cdot}002\,x$$
$$x\,(1{\cdot}002\,\sigma-1{\cdot}002)=\sigma\,(1{\cdot}002-1).$$
$$x=\frac{{\cdot}002}{1{\cdot}002}\left(\frac{\sigma}{\sigma-1}\right)$$
$$=\frac{1}{501}\,.\left(\frac{\sigma}{\sigma-1}\right)$$

7. Let A′ is the area of the cross-section. Let the sp. gr. of C is σ and W is the weight of the hydrometer.

$$W=(V-2A')\times\cdot8w=(V-3A')\times\cdot85\ w=(V-4A')\ \sigma w\ldots\text{I}$$

$$0\cdot85\ V-\cdot8V=2\cdot55\ A'-1\cdot6\ A'$$

$$\therefore\qquad 5V=95A'\ldots\ldots\ldots\ldots\text{II}$$

Also, $$\cdot8V-1\cdot6A'=V\sigma-4\sigma\ A'$$

or $$V\ (10\sigma-8)=(40\sigma-16)\ A'\ldots\ldots\ldots\text{III}$$

$$\therefore\qquad V\ (10\sigma-8)=(40\sigma-16)\frac{5V}{95}$$

$$190\ \sigma-152=40\ \sigma-16.$$

$$150\ \sigma=136$$

$$\sigma=\tfrac{136}{150}=\cdot906\ \text{nearly.}$$

8. Let σ_1 and σ_2 be the specific gravities of the two liquids.

From the condition of floating bodies

$$(8+2)\ \text{oz}=V\ .\ \sigma_1\ldots\ldots\ldots\text{I}$$

$$(8+5)\ \text{oz}=V\ .\ \sigma_2\ldots\ldots\ldots\text{II}$$

Dividing I by II

$$\frac{\sigma_1}{\sigma_2}=\frac{10}{13}.$$

9. Let the specific gravities of the two liquids be σ_1 and σ_2. Hence from the condition of floating bodies

$$4\tfrac{3}{4}+2=V\ .\ \sigma_1$$

$$\tfrac{27}{4}=V\ .\ \sigma_1\ldots\ldots\ldots\text{I}$$

Also, $$4\tfrac{3}{4}+2\tfrac{3}{8}=V\ .\ \sigma_2$$

$$\tfrac{57}{8}=V\ .\ \sigma_2\ldots\ldots\ldots\text{II}$$

Dividing I by II

$$\frac{\sigma_1}{\sigma_2}=\frac{27}{4}\cdot\frac{8}{57}$$

$$=\frac{54}{57}=\frac{18}{19}$$

10. Let x oz is the required weight to sink it to the fixed mark.

$$3\tfrac{3}{4}+1\tfrac{3}{4}=V\ .\ \text{I}\ .\ W$$

$$\tfrac{11}{2}=V\ .\ W\ldots\ldots\ldots\text{I}$$

$$3\tfrac{3}{4}+x=V\,.\,(2\cdot 2)\,.\,W\ldots II$$

$$\tfrac{15}{4}+x=2\cdot 2\times\tfrac{11}{5}$$

$$x=\frac{11}{2}\cdot\frac{11}{5}-\frac{15}{4}$$

$$=\frac{121}{10}-\frac{15}{4}=\frac{242-75}{20}=\frac{167}{20}$$

$$=8\tfrac{7}{20}.$$

11. Let W is the weight of the hydrometer, W_1 is the weight of solid and σ is the sp. gr.

Therefore,

$$W+W_1+12=W+W_1\left(1-\frac{1}{\sigma}\right)+16=W+22$$

$$\therefore\quad W_1=10$$

$$10\left(1-\frac{1}{\sigma}\right)+16=22$$

$$10-\frac{10}{\sigma}=6$$

$$\frac{10}{\sigma}=4$$

$$\sigma=\tfrac{10}{4}=2\cdot 5.$$

12. Let W is the weight of the hydrometer, W_1 is the wt. of the substance and σ is the sp. gr. Therefore,

$$W+1250=W+W_1+530=W+W_1\left(1-\frac{1}{\sigma}\right)+620$$

$$\therefore\quad W_1=720.$$

Also $$1250=720\left(1-\frac{1}{\sigma}\right)+620$$

or $$720-\frac{720}{\sigma}=630$$

$$\frac{720}{\sigma}=90$$

$$\therefore\quad \sigma=\tfrac{720}{90}=8.$$

13. Let x oz. is the required wt. placed in the lower pan.

$$2=x\left(1-\tfrac{1}{8}\right)$$

$$x=\tfrac{16}{7}=2\tfrac{2}{7} \text{ oz.}$$

14. Let W be the weight of the hydrometer, where V, V′ are the volumes of the immersed and unimmersed portions. Let σ_1 be the marked specific gravity, hence,

$$W=V\sigma_1 \ldots\ldots\ldots\ldots \mathrm{I}$$

When it is used in air, it sinks to the same level in a liquid of specific gravity σ_2.

$$\therefore \quad W=V\sigma_2+V'\sigma$$

$$\therefore \quad V\sigma_1=V\sigma_2+V'\sigma$$

$$\sigma_1=\sigma_2+\frac{V'}{V}\sigma$$

$$\therefore \quad \sigma'=\frac{V'}{V}\sigma$$

or $$\frac{\sigma'}{\sigma}=\frac{V'}{V}.$$

15. Let W and V be the original weight and volume, and W′, V′ the weight and volume of the part chipped off.

Let xA be the volume out of the liquid down to the mark which properly corresponded to a density α'. Then

$$W=(V-Ax)\alpha', \text{ also } W-W'=(V-V'-Ax)\alpha$$

$$\therefore \quad \frac{W}{\alpha'}-\frac{W-W'}{\alpha}=V' \ldots\ldots\ldots\ldots \mathrm{I}$$

Similarly,

$$\frac{W}{\beta'}-\frac{W-W'}{\beta}=V' \ldots\ldots\ldots\ldots \mathrm{II}$$

Also, $$\frac{W}{\gamma'}-\frac{W-W'}{\gamma}=V' \ldots\ldots\ldots\ldots \mathrm{III}$$

Hence from I and II

$$W\left[\frac{1}{\alpha'}-\frac{1}{\alpha}-\frac{1}{\beta'}+\frac{1}{\beta}\right]+W'\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)=0 \ldots\ldots \mathrm{IV}.$$

From I and III

$$W\left[\frac{1}{\alpha'}-\frac{1}{\alpha}-\frac{1}{\gamma'}+\frac{1}{\gamma}\right]+W'\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)=0\ldots\ldots V$$

Hence from IV and V

$$\left(\frac{1}{\alpha'}-\frac{1}{\alpha}-\frac{1}{\beta'}+\frac{1}{\beta}\right)\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$$

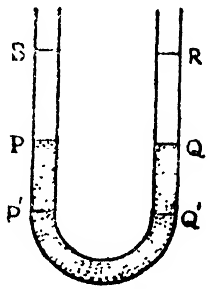
$$=\left(\frac{1}{\alpha'}-\frac{1}{\alpha}-\frac{1}{\gamma'}+\frac{1}{\gamma}\right)\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$\therefore\quad \frac{1}{\alpha'\beta}-\frac{1}{\alpha\beta'}=\frac{1}{\gamma}\left(\frac{1}{\alpha'}-\frac{1}{\beta'}\right)-\frac{1}{\gamma}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$\therefore\quad \gamma'=\frac{\alpha'\beta'\gamma(\alpha-\beta)}{\gamma(\alpha\beta'-\alpha'\beta)+\alpha\beta(\alpha'-\beta')}$$

$$\gamma=\frac{\gamma'\alpha\beta(\alpha-\beta)}{\gamma'(\alpha'\beta-\alpha\beta')+\alpha'\beta(\alpha-\beta)}$$

EXAMPLES XX



1. Initially the level of Mercury is at P and Q. Let the level of Mercury rises to R i. e. QR=1″, and the level of Mercury falls one inch in the other leg, so that level of Mercury is two inches below the former i. e. at P′. Let x is the required depth of water.

For equilibrium, pressure at P′ and Q′ must be the same.

$$2\times 13{\cdot}6\times w=x\times w$$

$$x=27{\cdot}2.$$

2. Let the water rises x inches in one leg when the other leg is filled with oil.

For equilibrium

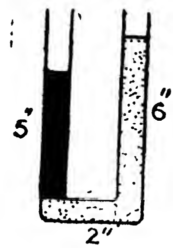
$$2x\,w=(x+4)\times\frac{2}{3}\,w.$$

$$2x=\frac{2x}{3}+\frac{8}{3}$$

$$\frac{4}{3}\,x=\frac{8}{3}$$

$$x=2.$$

Hence the amount of oil poured in $=x+4=2+4=6$ inches.



3. At the bottom of vertical tube pressure are equal.

$$5\cdot\tfrac{4}{5}=4.1$$

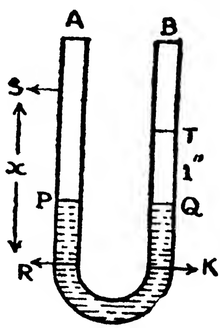
$$4=4.$$

Because there is 4 inches water in one tube, 5" oil in the other tube.

4. Let the specific gravity of the liquid is σ. The function of the Mercury and liquid in one leg is one inch below the level of the Mercury in the other, and 8 inches below the level of the liquid.

$$8\times\sigma=1\times13{\cdot}6$$

$$\sigma=1{\cdot}7.$$



5. Cross-section of A tube $=1$ sq. inch

Cross-section of B tube $={\cdot}1$ sq. inch.

Let the levels of Mercury in the two limbs be P and Q initially so that P and Q lie in the same horizontal plane. Also, let the level of Mercury rise from Q to T by 1" and that in left, fall from P to R when the water has been poured in left limb upto S. Then,

$$PR\times1=1\times{\cdot}1$$

$$PR={\cdot}1=KQ.$$

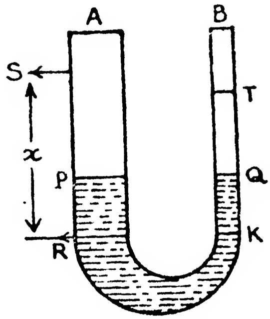
If now, K be a point in the right limb in the same horizontal level as R, then

$$11+xg=11+1{\cdot}1\times 13{\cdot}596\times g$$

$$x=1{\cdot}1\times 13{\cdot}596$$

Since the pressure at R and K; being in the same horizontal plane are equal.

$$\therefore \quad x=14{\cdot}9556 \text{ inch.}$$



6. Cross-section of A=2 sq. cm.

Cross-section of B=1 sq. cm.

$$\therefore \quad x\,.\,2=52$$

$$x=26 \text{ cm.}$$

Let the surface in A is lowered by y cm. It is raised $2y$ in the other, so that the difference between the levels is $3x$.

$$\therefore \quad 26=3x\times 13\,65$$

$$x=\frac{26}{3\times 13{\cdot}65}=\frac{40}{63} \text{ cms.}$$

EXAMPLES XXI

1. Let the height be h in oil barometer. Since the pressure at the same place are equal in both barometers, therefore

$${\cdot}9\times h=77{\cdot}4\times 13{\cdot}596$$

$$\therefore \quad h=\frac{77{\cdot}4\times 13{\cdot}596}{{\cdot}9}$$

$$=1169{\cdot}256 \text{ cms.}$$

2. Required thrust on the circular disc.

$$=\pi\,.\,(7)^2\times(5000+1033) \text{ grammes wt.}$$

$$=\tfrac{22}{7}\times 49\times 6033$$

$$=154\times 6033$$

$$=929082 \text{ grammes wt.}$$

3. Let the specific gravity of Glycerine is σ.

$$\therefore \qquad 26\,\sigma = 13\cdot 6 \times \tfrac{30}{12}$$

$$\sigma = \frac{13\cdot 6 \times 30}{26 \times 12} = 1\frac{4}{13}.$$

Weight of bullet + actual height × section of tube × Sw

= true height × section of tube × Sw

where S = sp. gr. of Mercury. Therefore

wt. of bullet = section of tube × x × Sw.

$\therefore$ actual height = true height − x.

4. The cistern level falls through a distance

$$= \frac{2\cdot 5 \times 1^2}{(4\cdot 5)^2} = \frac{10}{81} \text{ cm.}$$

$$\therefore \quad \text{Alteration in height} = 2\cdot 5 + \frac{10}{81} = 2\cdot 623 \text{ cm.}$$

5. The surface in the cistern falls a distance

$$= \frac{(\frac{1}{6})^2 \times 1}{(\frac{3}{2})^2} = \frac{1}{81} \text{ inch.}$$

$$\therefore \quad \text{Alteration} = 1 + \frac{1}{81} = 1\,\frac{1}{81} \text{ inch.}$$

EXAMPLES XXII

1. Let the specific gravity of air at standard pressure is σ. Hence

$$\sigma = \frac{760}{700} \times \cdot 0019$$

$$\sigma = \cdot 001292.$$

2. New weight $= 310 \times \dfrac{30\cdot 23}{}$

$$= 318\,\frac{4}{19} \text{ grs. wt.}$$

In areas in wt. $= 318\frac{4}{19} - 310$

$$= 8\frac{4}{19} \text{ grs. wt.}$$

3. Let x feet be the required depth.

Then from Boyle's law

$$PV = \text{Const.}$$

$$(33+10)\times 3 = (33+x) \cdot 2$$

$$129 = 66 + 2x$$

$$\therefore \quad 2x = 63$$

$$x = \frac{63}{2} = 31\frac{1}{2} \text{ feet.}$$

4. Let x be the required depth from the surface of water inside the tumbler. Hence the pressure of air there is due to a depth $(x+h)$ of water.

Hence from Boyle's law

$$V \times h = \frac{V}{3}(x+h)$$

$$h = \frac{x}{3} + \frac{h}{3}$$

$$\frac{2h}{3} = x$$

$$\therefore \quad x = 2h.$$

In the case of the conical wine-glass, since volumes of similar cones are as the cubes of their heights, the new volume of the air

$$= \frac{1}{23} \times \text{original volume}$$

Hence $$h \times 1 = (x+h) \times \frac{1}{23}$$

$$\therefore \quad x = 7h.$$

5. Let the height of the water barometer be h.

Hence from Boyle's law

$$V \times h = \frac{V}{2}(h + 32\cdot75)$$

$$h=\frac{h}{2}+\frac{1}{2}(32{\cdot}75)$$

$$\frac{h}{2}=\frac{1}{2}(32{\cdot}75)$$

$$\therefore \qquad h=32{\cdot}75 \text{ feet.}$$

6. Initially the level of the water outside and inside was the same. Let the tube be raised x cm.

Then the level of the water inside is

$$=x+25-50$$

$$=x-25 \text{ cms. above that outside.}$$

Hence the pressure of the contained air

$$=76-(x-25)$$

$$=(101-x) \text{ due to that of water.}$$

Hence from Boyle's law

$$25\times 76=50\,(101-x)$$

$$\therefore \qquad x=63.$$

7. Hence the pressure on one side

$$=5\times\tfrac{3}{4}\times 15 \text{ lbs. per sq. inch.}$$

$$=56\tfrac{1}{4} \text{ lbs. per sq. inch.}$$

Pressure on the other side

$$=\frac{3}{2}\times 15=\frac{45}{2}$$

$$=22\tfrac{1}{2} \text{ lbs. per sq. inch.}$$

When there is equilibrium let the piston be distant x inches from the centre.

$$\frac{5\times 15}{1+\frac{x}{12}}=\frac{1\times 15}{1-\frac{x}{12}}$$

$$5\left(1-\frac{x}{12}\right)=1+\frac{x}{12}$$

$$4=\frac{6x}{12}$$

$$x=8''.$$

9. Because the pressure of the atmosphere decreases the coal-gas expands till it again displaces its own weight of air and hence floats.

If it had been quite full originally it could no longer expand and thus as it no longer displaces as much as its own weight of air, it would sink.

10. Suppose x be the part of the weight that must be counter-passed.

Pressure equivalent to 1 inch of water

$$\text{i. e.}=\frac{1}{12}\,w.$$

Therefore,

$$60-x=\pi\left(1\frac{1}{4}\right)^2\times\frac{1}{12}\,w$$

$$x=60-\pi\,.\,\frac{25}{16}\times\frac{62\cdot 5}{12}$$

$$=34\cdot 4$$

11. When the level of water inside the bottle is 11 feet, then the volume of the air is given by

$$=\frac{33}{33+11}=\frac{3}{4}\text{ of its original volume.}$$

Weight of the displaced water

$$=\tfrac{3}{4}\times w=15\text{ oz.}$$

We know that

wt. of bottle $+5$ oz $=$ weight of water originally displaced.

$$=20\text{ oz.}$$

$\therefore$ Weight of bottle $=15$ oz.

Hence when lowered, the weight of bottle is just balanced by the weight of the displaced water. Hence it will just float.

When lowered the air becomes decreased in volume, and so the upward thrust of the displaced water becomes lessened and the bottle sinks.

Similarly if it be raised the upward thrust is increased and the bottle rises.

12. The volume of the air decreased from a length a of the cylinder to a length $(a-k)$.

Hence the pressure of the air is increased to $\left(\frac{ah}{a-k}\right)w$

Hence the pressure at the base is

$$(a+k)\ w+\left(\frac{ah}{a-k}\right)w=2\ (a+h)\ w$$

$$a^2-k^2+ah=2a^2-2ak-2hk+2\ ha$$

$$\therefore\quad k^2-2k\ (h+a)+ha+a^2=0$$

$$\therefore\quad k=h+a\pm\sqrt{h^2+ah}$$

The upper sign is inadmissible ; since it makes

$k > a$; which is impossible

$$\therefore\quad k=h+a-\sqrt{h^2+ah}.$$

13. Suppose σ is the sp. gr. of iron and h is the height of water barometer.

The pressure of the contained air is given by

$$=\frac{7h}{6}\ w$$

Hence

$$\frac{2}{3}\ \sigma w=\left(\frac{7hw}{6}-hw\right)=\frac{hw}{6}$$

$$\sigma=\frac{h}{4}\ \ldots\ldots\ldots\ldots\ldots \mathrm{I}$$

The final pressure of air

$$=\frac{7\ hw}{7-1\frac{36}{41}}$$

$$=\frac{287\ hw}{210}$$

$$=\frac{41\ hw}{30}.$$

$$\therefore \quad \frac{2}{3}\sigma w+6w=\left(\frac{41\ hw}{30}-hw\right)=\frac{11\ hw}{30} \quad \ldots\ldots\ldots\ldots \text{II}$$

$$\therefore \quad h=30 \text{ and } \sigma=\frac{h}{4}=7\frac{1}{2}.$$

14. The pressure of the air inside the hollow cylinder in water

$$=k-(h-x)$$
$$=x+k-h$$

From Boyle's law

$$x\,(x+k-h+\text{H})=h\ .\ \text{H}.$$
$$x^2+(\text{H}+k-h)\,x=\text{H}\ .\ h.$$

which is the required equation.

15. Suppose the mercury have risen to a height x inches. The pressure of the air inside is equal to that of water at a depth $4\frac{7}{32}-x$.

Hence from Boyle's law

$$\left(4\frac{7}{32}-x\right)\left(4\frac{7}{32}-x+30\right)=4\frac{7}{32}\times 30.$$

Solving we get

$$x=\frac{15}{32}.$$

16 Let the height occupied by air is h, so that volume of the air is given by $\left(\frac{h}{4}\right)^3\times$original volume

Hence from Boyle's law

$$\left(\frac{h}{4}\right)^3(34+34)=34\times 1$$

$$h=\frac{4}{(2)^{\frac{1}{3}}}.$$

17. Let the height of the cone is a. In the first case, suppose x is the length of the axis occupied by the compressed air. In the second case the whole of the cone is immersed.

$$\therefore \quad \frac{W}{mW+W}=\frac{x^3}{a^3}.$$

$$a^3=(1+m)\,x^3.$$

Comparing the pressures at the common surface,

$$h+x=\frac{a^3\,h}{x^3}$$

$$\frac{h+x}{h}=\frac{a^3}{x^3}=1+m$$

$$1+\frac{x}{h}=1+m$$

$$\frac{x}{h}=m.$$

Hence $\dfrac{a^3}{m^3\,h^3}=1+m$ or $\dfrac{a}{h}=m\,(1+m)^{\frac{1}{3}}$ $\therefore$ $a=mh\,(1+m)^{\frac{1}{3}}$.

18. When the vessel is pushed into water, let the depth of mouth is 13 feet, also suppose the air inside the cylinder occupy a length x. Hence

$$x\,(33+13-8+x)=8\times 33$$

Solving, we get $x=6$.

Let A be the internal cross-section, and therefore $\frac{3}{4}$ A the cross-section of the iron.

The volume of the water displaced

$$=\tfrac{3}{4}\,A\times 8+A\times 6=12\,A$$

$\therefore$ thrust of the water$=12\,Aw$

The weight of the cylinder

$$=\frac{3A}{4}\times 8\times 2w=12\,Aw.$$

The cylinder is just in equilibrium. If this cylinder is lowered further, the cylinder will sink.

19. Suppose π be the atmospheric pressure. If the piston sinks 2 ft., the pressure of the air under it$=\frac{5}{3}\pi$.

Let A be the area of cross-section of the piston. Hence

$$A\left[\frac{5\pi}{3}-\pi\right]=30$$

$$\frac{2\,A\pi}{3}=30\ldots\ldots\ldots\ldots\text{I}$$

When the piston has sunk another 2 feet, the pressure of the contained air is given by

$$=5\pi.$$

Let x be the force applied to the piston in lbs. wt. Therefore,

$$A\,(5\pi-\pi)=x+30$$
$$4\,A\pi=x+30$$
$$4\times45=x+30$$
$$x=180-30$$
$$=150.$$

20. Suppose the air occupy a length x in the cylinder at atmospheric pressure.

The pressure of the air

$$=\pi x$$

Also $\pi\,.\,(\frac{1}{2})^2\,(\pi x-\pi)=62{\cdot}5\pi+\pi\,(\frac{1}{2})^2.\,3w\ldots\ldots\ldots\ldots\text{I}$

When the sphere is immersed in water, let the piston sink to a depth of y feet. Therefore, new pressure of the air

$$=\frac{\pi x}{1-y}$$

The effect of the immersion is to increase the thrust on the piston by the weight of a volume of water equal to that of the sphere i. e. by

$$\frac{4}{3}\pi\left(\frac{3}{8}\right)^3.\,w$$

Therefore

$$\pi \cdot \left(\frac{1}{2}\right)^2 \left[\frac{\pi x}{1-y} - \pi\right] = 62{\cdot}5\ \pi + \pi \left(\frac{1}{2}\right)^2 . 3w + \frac{4}{3}\pi\left(\frac{3}{8}\right)^3 w \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

If h is the height of the water barometer in feet, then, $\pi = wh$ and $w = 62{\cdot}5$.

From I,

$$\pi \cdot \tfrac{1}{4} \cdot wh\,(x-1) = w\pi + \pi \cdot \tfrac{1}{4} \cdot 3w$$

$$h\,(x-1) = 7 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From II,

$$\pi \cdot \frac{1}{4} \cdot wh \cdot \left(\frac{x}{1-y} - 1\right) = w\pi + \pi\frac{1}{4} \cdot 3w + \frac{4}{3}\pi \cdot \frac{27}{64{\cdot}8} w.$$

or $$\frac{h}{4}\left(\frac{x}{1-y} - 1\right) = 1 + \frac{3}{4} + \frac{9}{128}$$

or $$h\left(\frac{x}{1-y} - 1\right) = \frac{233}{32} \ldots\ldots\ldots(4)$$

or $$h\,(x-1+y) = \frac{233}{32}(1-y)$$

or $$h\,(x-1) + hy = \frac{233}{32}(1-y)$$

or $$7 + hy = \frac{233}{32}(1-y)$$

or $$y\left(h + \frac{233}{32}\right) = \frac{233}{32} - 7$$

or $$y\,(32\,h + 233) = 9$$

$$\therefore \qquad y = \frac{9}{32\,h + 233}.$$

21. Suppose x be the distance from the centre at which the piston rests, so that the pressure of the air above and below it are

$$\frac{\pi\, a}{a+x} \text{ and } \frac{\pi\, a}{a-x}.$$

Hence

$$A\left[\frac{\pi a}{a-x}-\frac{\pi a}{a+x}\right]=W \sin \alpha$$

$$A . \pi a\left(\frac{a+x-a+x}{a^2-x^2}\right)=W \sin \alpha$$

$$W . \lambda . a . \frac{2x}{a^2-x^2}=W \sin \alpha$$

$$(2 \lambda a \operatorname{cosec} \alpha)\, x=a^2-x^2$$

$$\therefore \quad x^2+2\lambda a . \operatorname{cosec} \alpha . x-a^2=0$$

$$x=-\lambda a . \operatorname{cosec} \alpha+\sqrt{a^2+\lambda^2 a^2 \operatorname{cosec}^2 \alpha}$$

EXAMPLES XXIII

1. (i) Let the volume at 0°C is V, then

$$\frac{76\times V}{1}=\frac{100\times 80}{1+\alpha . 30}$$

$$\text{where } \alpha=\frac{1}{273}$$

$$76\times V=\frac{8000}{1+\frac{30}{273}}$$

$$V=\frac{8000\times 273}{303\times 76}$$

$$=94{\cdot}84.$$

(ii) We know that

$$100^\circ F=\frac{5}{9}(100-32)\,^\circ C=\frac{5\times 68}{9}\,^\circ C$$

$$=\frac{340}{9}\,^\circ C$$

3 atmosphere $=3\times 76$ cm. of Mercury

Let the required volume is V.

$$V\times 76=\frac{3\times 3\times 76}{1+\frac{1}{273}\times\frac{340}{9}}$$

$$\therefore \quad V=\frac{9\times 9\times 273}{2797}$$

$$=7{\cdot}90\ldots\text{cubic feet.}$$

2. Let V is the required volume.

Hence from Charle's law

$$\frac{V\times 51}{1+\alpha\,.\,16}=\frac{57\times 9}{1+\alpha\,.\,69} \text{ where } \alpha=\frac{1}{273}$$

or
$$\frac{V\times 51}{1+\frac{1}{273}\,.\,16}=\frac{57\times 9}{1+\frac{69}{273}}.$$

$$\frac{V\times 51}{289}=\frac{57\times 9}{342}$$

$$V=\frac{57\times 9\times 289}{51\times 342}.$$

$$=8\tfrac{1}{2} \text{ cubic inches.}$$

3. Let V is the required volume.

Then from Charle's law

$$\frac{V\times 54}{1+\alpha\,.\,t}=\frac{15\times 32}{1+\alpha\,.\,t'}$$

or
$$\frac{V\times 54}{1+\frac{78}{273}}=\frac{15\times 32}{1+\frac{39}{273}}$$

or
$$\frac{V\times 54}{351}=\frac{15\times 32}{312}$$

$$V=\frac{15\times 32\times 351}{54\times 312}$$

$$=10 \text{ cubic inches.}$$

4. Let the volume of air at the bottom is one cubic metre and V is the volume at the top of the mountain.

Therefore from Charle's law

$$\frac{400\times V}{1+\frac{31}{273}}=\frac{750\times 1}{1+\frac{7}{273}}$$

8. Initially the length of Mercury in the closed limb be x, and h be the height of the Mercury barometer. When the Mercury poured in, the difference between the levels of the ends of the Mercury is $8-2=6$ and the air has got a pressure $(6+h)$.

From Boyle's law

$$x \times h=(x-1) \times (6+h) \ldots\ldots\ldots \text{I}$$

When 11 inches Mercury added, the difference

$$=8+11-4=15$$

From Boyle's law

$$x \times h=(x-2) \times (15+h) \ldots\ldots\ldots \text{II}$$

Solving I and II, we get

$$x=6 \text{ and } h=30.$$

9. The difference between the readings of barometer

$$=762-700=62 \text{ mm.}$$

Therefore pressure is that due to 62 mm.

$\therefore$ Required pressure$=\frac{62}{10} \times 13{\cdot}596$ gram wt. per sq. cm.

$$=84{\cdot}2952 \text{ grammes wt.}$$

10. Case I

The air occupied length one inch and pressure one inch of Mercury.

Case II

Let h is the required height. Length of air$=1+31-29\frac{1}{2}$ i. e. $2\frac{1}{2}$ inches and pressure$=h-29\frac{1}{2}$.

From Boyle's law

$$2\tfrac{1}{2}\,(h-29\tfrac{1}{2})=1 \times 1$$

$$h=29\tfrac{1}{2}+\tfrac{2}{5}=29{\cdot}9$$

$$h=29{\cdot}9.$$

11. Suppose, length occupied by the air in the barometer be x and nx the length it would occupy at atmospheric pressure.

Therefore from Boyle's law

$$x \times (30-29{\cdot}8)=nx \times 30$$

$$n=\frac{\cdot 2}{30}=\frac{1}{150}.$$

12. If the surface of the Mercury be lowered y inches, then the air now occupies $(2+y)$ inches and the pressure is y inches of Mercury.

From Boyle's law

$$(2+y)\times y=\tfrac{1}{2}\times 30$$
$$y=3.$$

Let the height of a correct barometer be h.

Air occupied length $=30+2-x$.

$$=32-x$$

And the pressure $=(h-x)$ inches.

From Boyle's law

$$(32-x)\ (h-x)=\tfrac{1}{2}\times 30$$
$$h=x+\frac{15}{32-x}$$

13. Case I

Let x be the length of air and its pressure

$$=30{\cdot}4-29{\cdot}8$$
$$={\cdot}6 \text{ inches.}$$

Case II

Length of the air $=x+{\cdot}4$ and its pressure

$$=29{\cdot}8-29{\cdot}4$$
$$={\cdot}4 \text{ inches.}$$

From Boyle's law

$$x\times{\cdot}6=(x+{\cdot}4)\times{\cdot}4$$
$$x={\cdot}8 \text{ inches.}$$

When the faulty barometer is at 29,

the air occupies $=29{\cdot}8+{\cdot}8-29$

$$=1{\cdot}6 \text{ inches.}$$

Let the true height is h, the pressure is $(h-29)$.

From Boyle's law

$$1{\cdot}6\times(h-29)={\cdot}8\times{\cdot}6$$

$$h-29=\frac{\cdot 8\times\cdot 6}{1\cdot 6}=\cdot 3$$

$$h=29\cdot 9.$$

14. The length of the air $=a$ inches and its pressure $=(c-b)$ inches.

When the apparent reading is d inches.

The length of the air $=a+b-d$ and its pressure $=(x-d)$ inches

where x is the required ht.

From Boyle's law

$$(a+b-d)\ (x-d)=a\ (c-b)$$

$$x=d+\frac{a\ (c-b)}{a+b-d}.$$

EXAMPLES XXV

1. Let b be the height of the ball. At a depth a, let x be the length of the ball occupied by the air, and let h be the height of the water barometer.

Hence we know that $x^2+(a+h)\ x-h\ .\ b=0$

where $a=80$, $b=6$ and $h=33\frac{1}{3}$.

$\therefore\quad x^2+(80+33\frac{1}{3})x-33\frac{1}{3}\times 6=0$

or $\quad 3\,x^2+340\,x-600=0$

$$x=\frac{-170+10\sqrt{307}}{3}$$

The pressure is given by $\dfrac{x+a+h}{h}=\dfrac{17+\sqrt{307}}{10}$

$=3\cdot 45$ atmospheres.

2. The ball being always kept full of air, the depth of the lowest point is given by

$$\frac{x+h}{h}=\frac{31}{30}$$

$$\therefore\quad x=\frac{h}{30}=\frac{1}{30}\times 13\frac{1}{2}\times\frac{30}{12}$$

$=1\frac{1}{8}$ feet.

3. We know that the rise in the pressure is equivalent to a rise in the water barometer, which is

$$=\frac{13\cdot6\times12\frac{1}{2}}{12}\text{ feet}$$

$$=\frac{136}{10}\times\frac{25}{2\times12}$$

$$=\frac{85}{6}=14\frac{1}{6}\text{ feet.}$$

4. We know that

$$x^2+x\,(a+h)-h\,b=0$$

or $\quad x\,[x+a+h]=h\,.\,b$..............I

We are given, $b=9$, $h=34$, $x+a=17$

Hence

$$x\,(17+34)=34\times9$$

$$x=\frac{34\times9}{51}=6.$$

$\therefore\quad a=11.$

$\therefore$ Depth of the bottom of the well $=20$ feet

Let V be the required volume.

From Boyle's law

$$(V+9\times25)\times34=225\,(34+20)$$

$$(V+225)\,34=225\times54$$

$$V=\frac{225\times54}{34}-225$$

$$=\frac{2250}{17}$$

$$=132\frac{6}{17}\text{ cub. feet.}$$

5. Let V be the volume of air at atmospheric pressure.

The pressure of the contained air $=1\cdot02\times100+34$

$=136$ feet.

or $$\frac{400\text{ V}}{286}=\frac{750}{280}$$

$$\text{V}=\frac{750\times 286}{400\times 280}=\frac{429}{224}.$$

Therefore the weight of equal volumes at the top and bottom are in the ratio 224 : 429.

5. Initially the lengths occupied are $\frac{2l}{3}$ and $\frac{l}{3}$. If the temp. be raised, let the piston moves through a distance x. Let P be the original pressure.

The final pressures are

$$\text{P}\,(1+\alpha t)\,.\,\frac{\frac{2l}{3}}{\frac{2l}{3}+x} \text{ and P } \frac{\frac{l}{3}}{\frac{l}{3}-x}$$

Both these pressures must be equal

$$\text{P}\,.\,(1+\alpha t)\,\frac{2l}{2l+3x}=\text{P}\,.\,\frac{l}{l-3x}.$$

$$2\,(1+\alpha t)\,(l-3x)=2l+3x.$$

or $$3x\,(1+2+2\,\alpha t)=2l+2l\,\alpha\,t-2l$$

$$x=\frac{2\,l\,\alpha\,.\,t}{9+6\,\alpha t}.$$

6. Let the radius of the sphere is r, then the volume is $=\frac{4}{3}\,\pi r^3$. If the radius is doubled,

then the volume is $=\frac{4}{3}\,\pi\,(2r)^3=\left(\frac{4}{3}\,\pi r^3\right).\,2^3$

If p is the original pressure and p' the new pressure, therefore from Charle's law

$$\frac{p\,.\,\frac{4}{3}\,\pi r^3}{1}=\frac{p'\,.\,(\frac{4}{3}\,\pi r^3)\,.\,2^3}{1+\alpha\,.\,455}$$

$$p=\frac{p'\,.\,8}{1+\frac{455}{273}}$$

$$\text{or}\quad p=\frac{8\,p' \,.\, 273}{728}$$

$$\text{or}\quad p'=\frac{728}{8\times 273}\,p$$

$$\text{or}\quad p'=\frac{91}{273}\,p$$

$$\therefore\quad p'=\frac{p}{3}.$$

7. $P=13{\cdot}596\times 76\times 981$

$$V=\frac{1}{{\cdot}001}\text{ cub. cm.}=1000\text{ cub. cm.}$$

$$T=80+\frac{1}{{\cdot}00366}=\frac{129280}{366}$$

$$\therefore\quad \frac{P\,.\,V}{T}=\frac{13{\cdot}596\times 76\times 981\times 1000\times 366}{129280}$$

$$=2870000\text{ approx.}$$

8. Let the length of the cylinder be $2a$, so that the distances from the ends of the cylinder of the piston are $\sqrt{2}a$ and $(2a-\sqrt{2}a)$. Hence the pressures of the air are $\frac{\pi}{\sqrt{2}}$ and $\frac{\pi}{2-\sqrt{2}}$. Hence if A be the area of the piston and W its weight,

$$W=A\left(\frac{\pi}{2-\sqrt{2}}-\frac{\pi}{\sqrt{2}}\right)=\pi\,.\,A\text{.........I}$$

The temp. being raised from T to t_1 and t_2 respectively, the new pressure are given by

$$\frac{\pi a}{T}\Big/\frac{a}{t_1}=\pi\,.\,\frac{t_1}{T}\text{ and }\pi\,.\,\frac{t_2}{T}.$$

$$\therefore\quad \pi\left[\frac{t_1}{T}-\frac{}{T}\right]\times A=W=\pi\,.\,A.$$

$$\therefore\quad t_1-t_2=T.$$

EXAMPLES XXIV

1. The iron depresses the column of Mercury through a distance x in such a way that the weight of a length x of the tube of Mercury equals the weight of iron. We know that Mercury has nearly twice the specific gravity of iron that is why this depression is very small. While the air expands (i. e. forces the Mercury down) until its pressure differ from that of the external air by a quantity which is measured by the then height of the barometer.

2. Suppose the length of the original vacuum is x inches.

Length of the barometer tube$=(x+30)$ inches.

Length occupied by the air$=x+30-26$

$=(x+4)$ inches.

This air has a pressure$=30-26=4$ inches.

From Boyle's law

$$(x+4)\ \tfrac{1}{4}\times 4=\tfrac{1}{4}\times 30$$

$$x=\tfrac{7}{2}$$

$$\therefore \quad \text{Volume}=x\ .\ \frac{1}{4}=\frac{7}{2}\times\frac{1}{4}=\frac{7}{8}\ \text{cub. inch.}$$

3. If the Mercury depressed by x inches, air enclosed occupied $(1+x)$ inches length. The pressure of the air occupied is x.

Hence from Boyle's law

$$(1+x)\ .\ x=1\times 30.$$

$$x^2+x-30=0.$$

$$\therefore \quad x=5.$$

4. The pressure of the enclosed air$=29-28{\cdot}6$

$={\cdot}4$ ins.

The length occupied by the air$=33-28{\cdot}6$

$=4{\cdot}4.$

Let h be the true height. Then the air occupies $(33-29{\cdot}48)=3{\cdot}52$ inch at a pressure of $h-29{\cdot}48$.

From Boyle's law

$$3{\cdot}52\ (h-29{\cdot}48)={\cdot}4\times4{\cdot}4$$

$$h=29{\cdot}48+\frac{{\cdot}4\times4{\cdot}4}{3{\cdot}52}$$

$$=29{\cdot}98.$$

5. The pressure of the enclosed air $=28{\cdot}5-27$
$=1{\cdot}5$ ins. of Mercury.

The length occupied by the air $=36-27=9$ inches. Let h be the true height. Then the air occupies $36-30$ i. e. 6 ins. at a pressure $(h-30)$ inch.

$\therefore$ From Boyle's law

$$(h-30)\times6=9\times1{\cdot}5$$

$$h=30+\frac{4{\cdot}5}{2}$$

$$=32{\cdot}25.$$

6. Let x be the length of the tube of the faulty barometer. Then the air occupied is $(x-28)$ at pressure $30-28$ i. e. 2 ins. of Mercury.

In the second case the length $=x-14{\cdot}6$
and the pressure $=15-14{\cdot}6={\cdot}4$ ins.

Therefore from Boyle's law

$$(x-28)\times2=(x-14{\cdot}6)\times{\cdot}4$$

$$\therefore\quad x=31{\cdot}35$$

7. The difference between the levels of the surfaces of the Mercury $=38-4=34$ inches.

Hence the air is subjected to a pressure $=34+29\tfrac{1}{2}$
$=63\tfrac{1}{2}$ inches of Mercury.

Let x is the required length.

From Boyle's law

$$x\times29\tfrac{1}{2}=5\times63\tfrac{1}{2}$$

$$\therefore\quad x=\frac{5\times127}{59}=10\frac{45}{59}\text{ inches.}$$

Hence from Boyle's law

$$V\times 34=125\times 136$$
$$V=500 \text{ cub. feet.}$$

6. The new pressure is that due to $34+17=51$ feet of water.

$\therefore$ From Boyle's law

Original volume $\times$ 34 = final volume $\times$ 51

$$\therefore \quad \frac{\text{original volume}}{\text{final volume}}=\frac{51}{34}=\frac{3}{2}.$$

7. With the usual formula

$$x^2+x\,(a+h)-b\,h=0$$

Then, $x=8$; $b=10$.

Therefore,

$$64+8\,(a+h)-10\,h=0$$
$$32=h-4\,a\ldots\ldots\ldots\ldots\ldots\text{I}$$

When the air has been introduced,

The pressure $=(9+a+h)$ feet.

Let V be the total volume of the bell

$$\left(V+\frac{67}{440}V\right)\times h=\frac{9}{10}V\,.\,(9+a+h)$$

$$\therefore \quad 3a+27=\frac{37}{44}h\ldots\ldots\ldots\ldots\text{II}$$

Solving I and II, we get

$$h=33,\ a=\tfrac{1}{4}.$$

8. The volume of the air is reduced to $\frac{9}{10}$th of its initial volume ; hence the pressure is $\frac{10}{9}$ times the original. Let x be the required height.

From Boyle's law

$$x\times 13\cdot 5=\frac{10}{9}\times 33\frac{3}{4}$$

$$\therefore \quad x=\frac{25}{9}\text{ feet}=33\frac{1}{3}\text{ inch.}$$

The depth of the surface of the water in the bell

$$=(\tfrac{10}{9}-1)\times \text{ht.}$$

Water barometer$=3\frac{3}{4}$.

9. At time t, we know that

$$x^3+x\ (a+h)=b\ .\ h$$

If total air supplied in this time be V′, then

$$(a+h+b)\ \ V=(V+V')\ h$$

$$\therefore \qquad \frac{dV'}{dt}\,.\,h=V\,.\,\frac{da}{dt}.$$

$$\text{or} \qquad \frac{dV'}{dt}=\frac{V}{h}\,.\,\frac{da}{dt}=\text{const.}$$

10. The pressure of air inside the bell

$$=\left(h+nh-\frac{h}{20}\right) \text{ due to a depth of water.}$$

Let the bell contains air whose volume is V_1.
From Boyle's law

$$\frac{4}{5}V\left(h+nh-\frac{h}{20}\right)=V_1\times h$$

$$V=\frac{4}{5}\left(n+\frac{19}{20}\right)V.$$

11. With the usual relation

$$x^2+(a+h)\ x=h\ b.$$

$$x+a+h=\frac{h\ b}{x}.$$

$$a=\frac{h\ b}{x}-h-x$$

Hence substituting the values

$$\frac{hb}{\frac{2}{3}b}-h-\frac{2}{3}b=3\frac{1}{3}\left[\frac{hb}{\frac{3}{4}b}-h-\frac{3}{4}b\right]$$

$$\text{or} \qquad \frac{3}{2}h-h-\frac{2}{3}b=\frac{10}{3}\left[\frac{4}{3}h-h-\frac{3}{4}b\right]$$

or $$\frac{1}{2}h-\frac{2}{3}b=\frac{10}{3}\left[\frac{1}{3}h-\frac{3}{4}b\right]=\frac{10}{9}h-\frac{5}{2}b$$

or $$\frac{10}{9}h-\frac{1}{2}h=\frac{5}{2}b-\frac{2}{3}b$$

$$\therefore \quad b=\frac{h}{3}$$

$\therefore$ height of the cylinder $=\frac{1}{3}$ (height of the water barometer).

12. The pressure of the air in the diving bell is equal to that at the surface of the water inside of the bell, and therefore is greater than that of water at the level of the top of the bell. Hence when the hole is made the air flows out.

13. The length x of the bell occupied by the air is given by the equation

$$x^2+x\,(a+h)-h\,b=0$$

where the symbol have usual significance

$$x^2+x\,(47+34)-34\times 10=0$$

or $$x^2+81\,x-340=0$$

$$(x+85)\;(x-4)=0$$

$$\therefore \quad x=4.$$

The pressure of the inside air $=4+47+34$

$=85$ ft. of water

$$\therefore \quad \text{density}=\frac{85}{34}\,.\,\sigma=\frac{5}{2}\sigma.$$

Let ρ is the sp. gr. of wood

$$\therefore \quad \rho\,.\,1=\tfrac{1}{2}\,.\,1+\tfrac{1}{2}\,\sigma$$

$$\rho=\frac{1+\sigma}{2}.$$

Let y be the fraction of wood immersed in water

$$y\,.\,1+(1-y)\frac{5\sigma}{2}=1\,.\left(\frac{1+\sigma}{2}\right)$$

$$y\left(1-\frac{5\sigma}{2}\right)=\frac{1+\sigma}{2}-\frac{5\sigma}{2}=\frac{1-4\sigma}{2}$$

$$y=\left(1-\frac{4\sigma}{2}\right)\left(1-\frac{5\sigma}{2}\right)^{-1}$$

$$=\left(1-\frac{4\sigma}{2}\right)\left(1+\frac{5\sigma}{2}\right) \text{ approx.}$$

$$y=\frac{2-3\sigma}{4} \text{ nearly.}$$

14. The air occupied inside the bell $=16-4=12$ feet.

Volume of the air $=\left(\frac{12}{16}\right)^3$ of its original volume

$=\frac{27}{64}$ of its original volume.

Let h be the height of the water barometer

From Boyle's law

$$\frac{27}{64}\left(12+33\frac{2}{9}+h\right)=1\ .h$$

$$\therefore\quad h=33 \text{ feet.}$$

15. The length x of the bell occupied by the air is given by the equation

$$x^2+x\ (a+h)-h\ .\ b=0$$

When $x=\frac{2b}{3}$

$$\frac{4b^2}{9}+\frac{2b}{3}\ (a+h)-h\ .b=0.$$

or $\qquad 4b+6a+6h-9h=0$

or $\qquad 4b+6a=3h$....................I

When a volume equivalent to a length $\frac{3b}{2}$ of atmospheric air has been compressed to a length $\frac{b}{2}$, let a' be the new depth of the top of the bell. Therefore Boyle's law gives

$$\frac{3b}{2}\ .\ h=\frac{b}{2}\left(\frac{b}{2}+a'+h\right)$$

$$\therefore\quad a'=2h-\frac{b}{2}.$$

Therefore,

$$a'-a=2h-\frac{b}{2}-\frac{1}{6}(3h-4b)$$

$$=\frac{3h}{2}+\frac{b}{6}$$

(which is the required result)

16. Let σ is the sp. gr. of mercury. The respective heights of the water barometer are $\frac{h\sigma}{12}$ and $\frac{h'\sigma}{12}$ feet.

(1) Let x be the length of the axis of the cone from its vertex to the surface of the water inside the cone.

Therefore,

$$\frac{h'\sigma}{12}=x+a+\frac{h\sigma}{12}$$

From Boyle's law

$$x^3 \cdot \frac{h'\sigma}{12}=b^3 \cdot \frac{h\sigma}{12}$$

$$\therefore \quad a=\frac{\sigma}{12}(h'-h)-\left(\frac{h}{h'}\right)^{\frac{1}{3}} . b.$$

(2) In this case :

$$\frac{h'\sigma}{12}=x+a+\frac{h\sigma}{12}$$

$$x \cdot \frac{h'\sigma}{12}=b \cdot \frac{h\sigma}{12}$$

$$\therefore \quad a=(h'-h)\frac{\sigma}{12}-b \cdot \left(\frac{h}{h'}\right).$$

17. With the usual equation

$$x^2+(a+h)\,x-h\,b=0.$$

Substituting the values

$$\left(\frac{a}{2}\right)^2+(h+\mathrm{H})\frac{a}{2}-h \cdot a=0.$$

$$\frac{a}{2}=\mathrm{H}-h \dots\dots\dots\dots\dots \mathrm{I}.$$

Let the bell be immersed further by a distance

Hence from Boyle's law

$$a\ (a+h+\text{H})=\frac{a}{2}\left(\frac{a}{2}+h+y+\text{H}\right)$$

$$\therefore \quad y=\frac{3a}{2}+\text{H}+h$$

$$=3\ (\text{H}-h)+\text{H}+h$$

$$=4\ \text{H}-2h.$$

18. With the usual Eqn.

$$x^2+x\ (a+h)-h\ .\ b=0$$

$$\text{or} \quad x^2+x\ (100+30)-30\times 10=0$$

$$\text{or} \quad x^2+130\ x=300$$

$$\therefore \quad x+65=\sqrt{4525}$$

When the temperature is lowered, let y be the length of the bell occupied by air. Then

$$\frac{(y+130)\ y}{1+\frac{15}{273}}=\frac{(x+130)\ x}{1+\frac{20}{273}}$$

$$y^2+130\ y=300\times\frac{288}{293}=294{\cdot}88$$

$$y+65=\sqrt{4519{\cdot}68}.$$

The tension of the chain are

$$\text{W}-\pi\ .\ 9\ .\ xw \text{ and } \text{W}-\pi\ .\ 9\ y\ .\ w$$

Therefore increase in tension

$$=\pi\ .\ 9\ .\ w\ (x-y)=9\pi\ .\ \frac{125}{2}\left[\sqrt{4525}-\sqrt{4519{\cdot}68}\ \right]$$

$=67$ lbs. wt. nearly.

19. Let the original tension $=\text{W}_1-\text{A}\ .\ w\ .\ x$.

Let V be the volume of the water drawn up so that $\text{V}w=\text{W}$. Then if y be the new length of the bell occupied by the air, its new volume is $\text{A}y-\text{V}$, and then

$$(\text{A}y-\text{V})\ (y+a+h)=\text{A}x\ (x+a+h)$$

$$\therefore \quad (y-x)\,A\,.\,(y+x+a+h)=V\,(y+a+h)$$

$$y-x=\frac{V}{A}\left(\frac{y+a+h}{y+x+a+h}\right)$$

Since V is small y and x are very nearly equal and the right hand side of this expression$=\frac{V}{A}\left(\frac{x+a+h}{2x+a+h}\right)$ nearly.

Hence increase of tension

$$=(W_1+W-Ayw)-(W_1-Axw)$$
$$=W-A\ (y-x)\,w$$
$$=W-Vw\,.\,\frac{x+a+h}{2x+a+h}$$
$$=W-W\,.\left(\frac{x+a+h}{2x+a+h}\right)$$
$$=W\,.\left(\frac{x}{2x+a+h}\right)$$

20. If x be the length occupied by the compressed air at temp. T, we have

$$x^2+x\,(h+d)-ah=0\text{.............I}$$

In the second case when the temp. has been raised by t'^0 suppose y is the length occupied by the compressed air. Let A denote the area of cross-section of the bell.

$$\frac{PV}{T}=K$$

$$Ax\,(h+d+x)=KT$$
$$Ay\,(h+d+y)=K\,(T+t')$$
$$\frac{y\,(h+d+y)}{x\,(h+d+x)}=\frac{T+t'}{T}=1+\frac{t'}{T}.$$

or $\quad (y^2-x^2)+(h+d)\,(y-x)=x\,(h+d+x)\,\frac{t'}{T}$

$$=\frac{ah\,.\,t'}{T}\text{ from I}$$

$$y-x=\frac{ah\,t'}{T}\cdot\frac{1}{(y+x+h+d)}$$
$$=\frac{aht'}{T\,(2x+h+d)}\text{ nearly..........II}$$

Now, if W' be the weight of the bell and T_1 and T_2 the tensions in the two cases, we have

$$T_1=W'-Axw$$
$$T_2=W'-Ayw$$
$$\therefore\quad T_2-T_1=Aw\,(y-x)$$

Since $W=Aw\,.\,a$

$$\therefore\quad T_2-T_1=-(y-x)\,\frac{W}{a}$$
$$=-\frac{W}{a}\cdot\frac{aht'}{T\,(2x+h+d)}\text{ from.........II}$$
$$=-\frac{Wht'}{T}\,\frac{1}{(2x+h+d)}$$

From I, we have

$$2x+h+d=\sqrt{(d+h)^2+4\,ah}$$
$$\therefore\quad T_2-T_1=-\frac{Wht'}{T}\cdot\frac{1}{\sqrt{(h+d)^2+4\,ah}}$$

Hence, the tension of the supporting chain is diminished by

$$=\frac{W\,.\,h\,(t_1-t)\,\alpha}{1+\alpha t}\cdot\frac{1}{\sqrt{(h+d)^2+4\,ah}}$$

21. The volume now occupied by the air is to the original volume as $x^3 : a^3$.

Hence by Boyle's law

$$x^3\,(x+d+h)=a^3\,.\,h.$$
$$x^4+x^3\,(h+d)=a^3\,.\,h\text{............I}$$

Let y be the length of the part of the bell occupied by the air when the temp. is raised. Therefore

$$\frac{y^3\,(y+d+h)}{1+\alpha\,(T+t)}=\frac{x^3\,(x+d+h)}{1+2T}=\frac{a^3\,h}{1+\alpha T}$$
$$\therefore\quad y^4+(d+h)\,y^3=\left(1+\frac{\alpha t}{1+T}\right)a^3\,h\text{.........II}$$

Subtract I from II

$$y^4-x^4+(d+h)(y^3-x^3)=\frac{\alpha t}{1+\alpha T}\cdot a^3 h.$$

$$(y-x)=\frac{a^3 h}{(y^2+x^2)(y+x)+(d+h)(y^2+xy+x^2)}\times\frac{\alpha t}{1+\alpha T}$$

Since α is small, y and x are nearly equal.

$$\therefore\quad y-x=\frac{a^3 h}{4x^3+3x^2(d+h)}\cdot\frac{\alpha t}{1+\alpha T}\text{ nearly.}$$

Hence the reqd. tension

$$=\left(W_1-V\cdot\frac{x^3}{a^3}w\right)-\left(W_1-V\cdot\frac{y^3}{a^3}w\right)$$

$$=\frac{Vw}{a^3}(y^3-x^3)$$

$$=\frac{W}{a^3}(y-x)(y^2+xy+x^2)$$

$$=\frac{W}{a^3}(y-x)\cdot 3x^2\text{ nearly}$$

$$=\frac{W}{a^3}\cdot 3x^2\cdot\frac{a^3 h}{4x^3+3x^2(d+h)}\cdot\frac{\alpha t}{1+\alpha T}$$

$$=\frac{3\,Wh}{4x+3d+3h}\cdot\alpha t\quad\text{nearly.}$$

22. If x be the length occupied by air originally, we have

$$x^2+x(a+h)=bh$$

or $$2x+a+h=\sqrt{(a+h)^2+4h}$$

If $(h+\delta h)$ be the new height of the water barometer, we have, A being the area of the cross-section of the bell,

$$P=A\cdot w\cdot\delta h$$

Corresponding change in x is given by

$$2x\cdot\delta x+(a+h)\,\delta x=(b-x)\,\delta h$$

$$\delta x=\frac{b-x}{2x+a+h}\cdot\delta h.$$

Hence increase in the tension $= -\text{A} \,.\, w \,.\, \delta x$

$$= -\text{P} \,.\, \left(\frac{b-x}{2x+a+h}\right)$$

$$= \frac{\text{P}}{2}\left[1 - \frac{a+h+2b}{2x+a+h}\right]$$

$$= \frac{\text{P}}{2}\left[1 - \frac{a+h+2b}{\sqrt{(a+h)^2+4bh}}\right]$$

EXAMPLES XXVI

1. Initial height $= 13{\cdot}6 \times 28$ inches $= \dfrac{13{\cdot}6 \times 28}{12}$ feet

$= 31{\cdot}73$ feet

Final height $= 13{\cdot}6 \times 31$ inches $= \dfrac{13{\cdot}6 \times 31}{12}$ feet

$= 35{\cdot}13$ feet.

Hence the height varies from $31{\cdot}73$ feet to $35{\cdot}13$ feet.

2. The height of the petroleum barometer would

$= \dfrac{33 \text{ ft. } 8 \text{ inches}}{{\cdot}8}$

$= 42$ feet 1 inch.

3. Height of the sea-water barometer

$= \dfrac{34 \text{ feet } 2 \text{ ins.}}{1{\cdot}025}$

$= \dfrac{200}{6}$ feet

$= 33$ feet 4 inches.

which is the greatest depth of the tank.

4. Let the area of the section of the barrel be A sq. feet. Then,

$$\text{A} \,.\, \text{I} \,.\, w = 10$$

The tension of the piston rod

$$= \text{A} \,.\, w \,.\, 24$$

Required work $=A\,.\,w\,.\,24\,.\,\frac{1}{3}$ ft. lbs.

$=10\times\frac{24}{3}$

$=80$ ft. lb.

5. The air which at atmospheric pressure occupied 4 inches will at the end of stroke occupy 6 inches and its pressure $=\frac{4}{6}\times$ atmospheric pressure which is equivalent to $\frac{4}{6}\times 32$ feet of water

$=2\frac{2}{15}$ feet.

The water would therefore rise in the barrel to a height

$=(32-2\frac{2}{15})$ feet.

$=29\frac{13}{15}$ feet.

which will therefore reach the pump, barrel.

6. The height x_1 at the end of the first stroke is given by the equation

$$ahc=(h-x_1)\,\{a\,(c-x_1)+Al\}$$

where symbols have usual meaning.

We have $c=16$, $h=32$, $A=16a$.

$\therefore\quad a\,.\,32\cdot16=(32-x_1)\,\{a\,(16-x_1)+16\,al\}$

$32\times16=(32-x_1)\,[16-x_1+16l]$

Let $x_1=16$;

$$32\times16=16\times16l$$

$$l=2 \text{ feet.}$$

If $l=1$; we have

$$32\times16=(32-x_1)\,(16-x_1+16)$$

$$=(32-x_1)^2$$

$$16\sqrt{2}=32-x_1$$

$$x_1=32-16\,(1\cdot414)$$

$$=9\cdot37 \text{ feet.}$$

7. Required force $=$ weight of $\dfrac{100}{144}\times200$ cubic feet of water.

$$=\frac{100\times200}{144}\times\frac{125}{2}$$

$=8680\frac{5}{9}$ lbs. weight.

8. Required force = wt. of $\frac{10}{144} \times 60$ cubic feet of water.

$$= \frac{10}{144} \times 60 \times 62{\cdot}5$$

$$= 260\tfrac{5}{12} \text{ lbs. wt.}$$

9. Force to raise the piston

$$= \text{weight of } \frac{\pi \,.\, 6^2}{4 \times 144} \times 20 \text{ cub. feet of water.}$$

$$= \frac{5\pi}{4} \times \frac{125}{2} \text{ lbs. wt}$$

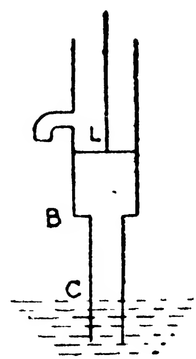
$$= \frac{625}{8}\,\pi \text{ lbs. wt.}$$

Force to depress the piston

$$= \frac{\pi}{4} \,.\, \frac{6^2}{144} \times 100 \times \frac{125}{2} \text{ lbs. wt.}$$

$$= \frac{3125}{8}\,\pi \text{ lbs. wt.}$$

10. Let A be the area of the piston, and w the weight of a unit volume of water.



The tension of the piston rod when the water is at L

$$= \text{A} \,.\, w \,.\, \text{CL}$$

Hence work done per stroke

$$= \text{Tension} \times \text{BL}$$

$$= \text{A} \,.\, w \,.\, \text{CL} \,.\, \text{BL}.$$

11. The force required in backward stroke

$$= \text{wt. of } \pi \,.\, 10^2 \,.\, 4 \,.\, 100 \text{ cub. cms. of water}$$

$$= 40\,\pi \text{ kilogrammes}$$

Force in forward stroke

$$= \text{wt. of } \pi \,.\, 10^2 \,.\, 60 \,.\, 100 \text{ cub. cms.}$$

$$= 600\,\pi \text{ kilogrammes wt.}$$

12. The height x_1 at the end of the first stroke is given by the equation

$$ahc = (h - x_1)\,\{a\,(c - x_1) + \text{A} \,.\, l\}$$

where the symbols have usual significance

Here $A=a$.

$$\therefore \quad c \, . \, h=(h-x_1)\,(c-x_1+l)$$

Also $$(h-x_1)\;\;c-x_1)=(h-x_2)\,(c-x_2+l)$$

We are given $x_2=2x_1$

$$\therefore \quad x_1^2-x_1\,(h+c+l)+hl=0\ldots\ldots\ldots\ldots\text{I}$$

$$3x_1^2-x_1\,(h+c+2l)+hl=0\ldots\ldots\ldots\ldots\text{II.}$$

Subtracting

$$2x_1^2-lx_1=0$$

$$x_1\,(2x_1-l)=0$$

$$\therefore \quad x_1=\frac{l}{2}.$$

From I ; $$l=(2h-2c).$$

$$\therefore \quad 2h=(c+l)+c.$$

$=$sum of the greatest and least distances of the piston

13. With the usual equation

$$ach=(h-x_1)\,[a\,(c-x_1)+A\,.\,l]$$

$$A=5a,\; h=34\;;\; l=10,\; x_1=c=10$$

$$34\times 10=(34-10)\,(10-10+5l)$$

$$\therefore \quad l=\frac{34\times 2}{24}$$

$$=2\,\frac{5}{6}\text{ feet.}$$

14. At the beginning of the stroke the air under the piston is at atmospheric pressure and of length 3 inches. At the end of the stroke its length is 12 inches.

$$\therefore \quad \text{Its pressure}=\frac{3}{12}\times 34\text{ ft. of water}$$

$$=\frac{17}{2}\text{ feet of water.}$$

Hence the greatest height

$$=34-8\tfrac{1}{2}$$

$$=25\tfrac{1}{2}\text{ feet.}$$

15. With the usual notation

$$ach=(h-x_1)\,[a\,(c-x_1)+A\,.\,l]$$

But $A=a\,.\,n.$

$$\therefore\quad ch=(h-x_1)\,[c-x_1+nl]\ldots\ldots\ldots\ldots\mathrm{I}$$

and
$$(h-x_1)\,(c-x_1)=(h-x_2)\,(c-x_2+nl)\ldots\ldots\ldots\mathrm{II}$$

We are given $x_2=c$

$$\therefore\quad (h-x_1)\,(c-x_1)=nl\,(h-c)\ldots\ldots\ldots\mathrm{III}$$

From I,
$$ch=(h-x_1)\left[\frac{nl\,(h-c)}{h-x_1}+nl\right]$$
$$=nl\,(h-x_1)+nl\,(h-c)$$

$$\therefore\quad x_1=2h-c-\frac{ch}{nl}.$$

Substitute in I,

$$ch=\left[c+\frac{ch}{nl}-h\right]\left[2c-2h+\frac{ch}{nl}+nl\right]$$

or
$$h^2\left[1-\frac{c}{nl}\right]\left[2-\frac{c}{nl}\right]-h\left[4c+nl-\frac{3c^2}{nl}\right]+c\,(2c+nl)=0$$

EXAMPLES XXVII

1. Density after the *n*th stroke$=\left(\dfrac{V}{V+V'}\right)^n.\,\rho.$

where V is the volume of the receiver and V′ be the volume of the cylinder between its higher and lower valves.

Hence
$$\left(\frac{V}{V+V'}\right)^4\rho=\frac{81}{256}\,\rho=\left(\frac{3}{4}\right)^4\rho.$$

i. e.
$$\frac{V}{V+V'}=\frac{3}{4}$$

or
$$4V=3V+3V'$$
$$V=3V'$$
$$\frac{V}{V'}=\frac{3}{1}$$

2. Here $V=36$, $V'=1\times4=4$.

$$\therefore \quad \frac{V}{V+V'}=\frac{36}{36+4}$$

$$=\frac{9}{10}$$

$\therefore$ Required ratio of pressure $=\left(\dfrac{9}{10}\right)^{8}$.

3. Case I.

$$\frac{V}{V+V'}=\frac{10}{10+1}$$

$$=\frac{10}{11}$$

Case II

$$\frac{V}{V+V'}=\frac{5}{5+1}=\frac{5}{6}$$

$\therefore$ Required ratio $=\left(\dfrac{10}{11}\right)^{3} \Big/ \left(\dfrac{5}{6}\right)^{3}$

$$=\frac{1728}{1331}$$

4. $\dfrac{V}{V+V'}=\dfrac{10}{10+1}=\dfrac{10}{11}$

Required ratio of pressure $=\dfrac{10^8}{11^8}$,

$$=\frac{100000000}{214358881}$$

5. Here $\dfrac{V}{V+V'}=\dfrac{6}{6+1}$

$$=\frac{6}{7}$$

$$\left(\frac{6}{7}\right)^{4}=\frac{1296}{2401}>\tfrac{1}{2}$$

$$\left(\frac{6}{7}\right)^{5}=\frac{7776}{16807}<\tfrac{1}{2}.$$

Also $$\left(\frac{6}{7}\right)^6=\frac{46656}{117649}>\tfrac{1}{3}$$

$$\left(\frac{6}{7}\right)^7=\frac{279936}{823543}>\tfrac{1}{3}$$

$$\left(\frac{6}{7}\right)^8=\frac{1679616}{5764801}<\tfrac{1}{3}$$

Hence 5 and 8 strokes are required,

6. Case I $$\frac{V}{V+V'}=\frac{12}{13}$$

Case II $$\frac{V}{V+V'}=\frac{6}{7}.$$

From the problem $$\left(\frac{6}{7}\right)^x=\left(\frac{12}{13}\right)^6=\frac{2985984}{4826809}$$ *

Also $$\left(\frac{6}{7}\right)^3=\frac{216}{343};\ \left(\frac{6}{7}\right)^4=\frac{1296}{2401}<\left(\frac{12}{13}\right)^6$$

Hence 4 strokes are required.

7. Here in the question

$$\left(\frac{V}{V+V'}\right)^{10}=\frac{20}{30}$$

$$\left(\frac{V}{V+V'}\right)^{30}=\left(\frac{2}{3}\right)^3=\frac{8}{27}$$

Height of the Mercury $=\frac{8}{27}\times 30$ inches $=8\frac{8}{9}$ inches.

8. Limiting pressure with the usual notation

$$\frac{ab}{(h+a)(h+b)}=\left(\frac{a}{h+a}\right)^2=\left(\frac{\frac{1}{4}}{6+\frac{1}{4}}\right)^2$$

$$=\frac{1}{625}$$

where $h=6$; $a=b=\frac{1}{4}$.

9. Let the numbers of strokes be x

Hence

$$\frac{1000+x \cdot 80}{1000}=4$$

$$100+8x=400$$

$$8x=300$$

$$x=\frac{75}{2}$$

$$=37\tfrac{1}{2}$$

$\therefore$ Between 37 and 38.

10. Here $V=\pi \cdot (\tfrac{1}{2})^2 \cdot 80$

$$=20\pi$$

and $V'=\pi \cdot (\tfrac{1}{2})^2 \cdot 8$

$$=2\pi.$$

Let the numbers of strokes b e x.

$\therefore$ $xV' \cdot \pi=V \cdot 2\pi$

$$x=2 \cdot \frac{V}{V'}=2 \cdot \frac{20\pi}{2\pi}$$

$$=20.$$

11. Let P be the greatest pressure exerted by the air. Therefore

$$5(P-15)=165$$

$$5P=165+75$$

$$=240$$

$$P=48.$$

Let the numbers of strokes be x.

$$\frac{10+x \cdot 1}{10}=\frac{48}{15}=\frac{16}{5}$$

or $50+5x=160$

$$5x=110$$

$$x=22.$$

12. A volume B of air at atmospheric pressure is condensed into a volume B—C, and its pressure becomes therefore $\frac{B}{B-C}$ atmospheres. When the air inside the receiver is at this or a greater pressure the valve in the receiver will thus not be opened.

13. We are given

$$V=8V'$$

Therefore $$\frac{V}{V+V'}=\frac{8}{9}$$

Density of the air at the end of IVth stroke$=(\frac{8}{9})^4 . \rho$.

This is also the density in the barrel at the beginning of the fifth stroke. When the upper valve open let a fraction x of the barrel has been described by the piston.

From Boyle's law

$$(\tfrac{8}{9})^4 . \rho . 1=\rho . (1-x)$$

$$\therefore \quad x=1-(\tfrac{8}{9})^4$$

$$=\frac{2465}{6561}$$

14. Let us assume ρ be the density of atmospheric air and let ρ' the density of the air in the receiver.

From Boyle's law

$$\rho'=\frac{1}{4}\rho .$$

15. If V be the volume of the speaking tube between the open end and the obstruction. But $V'=50$.

Hence from the condition of the problem,

$$\frac{V+30V'}{V}=\frac{V+30\times50}{V}=4$$

$$\text{or} \quad V+30\times50=4V$$

$$3V=30\times50$$

$$V=500$$

The section of the tube is 1 square inch.

length of the tube $=500$ inch.

$=41\frac{2}{3}$ feet.

16. Here $V=20V'$.

At the end of 20 strokes, density $=\frac{V+20V'}{V}\cdot\rho$

$$=\frac{20V'+20V'}{20V'}\cdot\rho$$

$$=2\rho.$$

But $\frac{V}{V+V'}=\frac{20}{21}$

Density at the end of 14 strokes or more $=\left(\frac{20}{21}\right)^{14}\times 2\rho.$

which is nearly $\frac{1}{2}$.

17. Let σ be the density in the barrel when the piston is at the highest point.

When the piston moves down, the volume of air C expands to volume $B-C'$.

$\therefore$ its density $=\frac{C\cdot\sigma}{B-C'}$.

If ρ be the density of the air in A, then for the valve from A to open we must have

$$\rho > \sigma\cdot\frac{C}{B-C'}\text{...............I}$$

In a similar way, the volume of air $B-C$ below the density is $\sigma\left(\frac{B-C}{C'}\right)$.

If ρ' be the density in the receiver, then

$$\sigma\cdot\frac{B-C}{C'} > \rho'\text{..................II}$$

Hence the required ratio.

18. Let the maximum pressure of the air in the receiver is x. When the piston is at the lowest point there is a volume V′ beneath it of pressure $x+p$, because the pressure ou

the conder ser-valve is just sufficient not to open it. When the piston is at the highest point the pressure of the air $=\dfrac{(x+p)V'}{V}$.

Hence $$(x+p)\frac{V'}{V}=\pi-p$$

$$x=\frac{V}{V'}(\pi-p)-p.$$

19. Let ρ_{n-1} be the density in the receiver and barrel at the end of $(n-1)^{th}$ stroke and ρ the initial density.

From Boyle's law

$$A\,\rho_{n-1}+C\,.\,\rho=(A+B)\,\rho_n$$

Therefore, $$\rho_n-\frac{C}{B}\rho=\frac{A}{A+B}\rho_{n-1}+\frac{C\rho}{A+B}-\frac{C\rho}{B}$$

$$=\frac{A}{A+B}\,.\,\rho_{n-1}-\frac{AC}{B(A+B)}\,.\,\rho$$

$$=\frac{A}{A+B}\left(\rho_{n-1}-\frac{C}{B}\rho\right)$$

Also $$\rho_{n-1}-\frac{C}{B}\,.\,\rho=\frac{A}{A+B}\left(\rho_{n-2}-\frac{C}{B}\rho\right)$$

$$\vdots$$

$$\rho_1-\frac{C}{B}\rho=\frac{A}{A+B}\left(\rho-\frac{C}{B}\,.\,\rho\right)$$

Henee, by multiplying

$$\rho_n-\frac{C}{B}\rho=\left(\frac{A}{A+B}\right)^n\left(\rho-\frac{C}{B}\,.\,\rho\right)$$

$$\frac{\rho_n}{\rho}=\frac{C}{B}+\left(1-\frac{C}{B}\right)\left(\frac{A}{A+B}\right)^n$$

EXAMPLES XXVIII

1. Required height $=13\cdot6\times30$

$=34$ feet.

2. Greatest height$=\frac{30\times 13{\cdot}6}{1{\cdot}5}$ inches.

$$=\frac{272}{12}\text{ feet}$$

$$=22\text{ feet }8\text{ inches.}$$

3. Siphon can work for a height equal to that of the Mercury-barometer which is 30 inches, and this is much less than 36 inches. That is why Siphon will not be able to remove all by this means.

4. Let the Siphon just stop working when the depth of the surface of water below the top of the vessel is x. If h be the height of water-barometer,

The pressure of the air$=\pi\cdot\frac{\frac{h}{4}}{x}=wh\cdot\frac{h}{4x}$.

The pressure at the bottom of the water

$$=\frac{wh^2}{4x}+w\,(h-x)$$

This must equal to wh.

$$\frac{h^2}{4x}-x=0$$

$$\therefore\quad x=\frac{h}{2}.$$

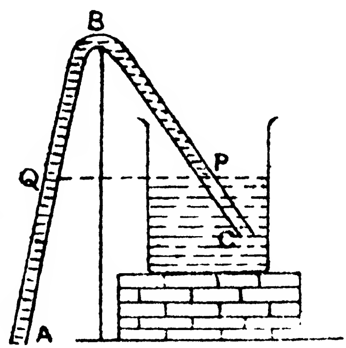
and depth by which the surface of the water is lowered

$$=\frac{h}{2}-\frac{h}{4}=\frac{h}{4}$$

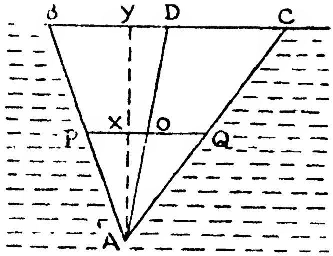
Therefore one-third of the water is removed.

5 If a hole be made in the shorter limb BC the flow ceases, the fluid below the hole falling back into the vessel.

If it be made in the longer arm below the level of Q the motion goes on, the liquid flowing out at this hole. If it be made in the longer arm above Q, the action of the instrument stops and the liquid flows back into the vessel.



EXAMPLES XXIX



1. Let ABC be a triangle wholly immersed in the liquid with its base BC in the surface. Let D be the middle point of BC and O that of AD, so that O is the centre of pressure and PQ the horizontal straight line drawn through the centre of pressure. Then P and Q will be the middle points of AB and AC respectively.

Now, if AY is the perp. drawn from A to BC cutting PQ in X, then

the thrust on the $\triangle$ ABC$=W(\frac{1}{2}BC \cdot AY) \cdot \frac{1}{3}AY.$

$$=\frac{1}{6}w\,BC \cdot AY^2$$

Thrust on the $\triangle$ APQ$=w(\frac{1}{2}PQ \cdot AX)(XY+\frac{1}{3}AX).$

$$=w(\tfrac{1}{2}\cdot\tfrac{1}{2}BC\cdot\tfrac{1}{2}AY)(\tfrac{1}{2}AY+\tfrac{1}{3}\cdot\tfrac{1}{2}AY)$$

$$=\tfrac{1}{12}w\cdot BC\cdot AY^2$$

$$=\tfrac{1}{2}\text{ (the thrust on the }\triangle\text{ ABC)}$$

$$=\text{The thrust on the portion BCQP.}$$

2. Let ABCD be the symmetrical section of the box, AB being the lid, and ABC the string whose tension is T.

Taking moment about A, we get

$$T\cdot AB=AB^2\times\frac{AB}{2}w\times\text{depth of C. P. of lid below A.}$$

$$T=\frac{w}{2}\cdot AB^2\times\frac{2}{3}AB=\frac{1}{3}WAB^3.$$

$$=\tfrac{1}{3}\text{ (weight of the water)}$$

3. Suppose $a, a, \frac{a}{2}$ be the lengths of the edges of the box and w_1 its weight per sq. foot.

Taking moments about the hinged edge

$$w_1a^2\cdot\frac{a}{2}+4w_1\frac{a^2}{2}\cdot\frac{a}{4}$$

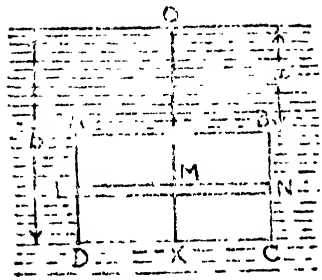
$=$Moment of the pressure on the outside face
$-$Moment of the weight of the water

$$=a^2 \cdot \frac{a}{2} \cdot w \cdot \frac{2a}{3} - w \cdot \frac{1}{2} a^3 \cdot \frac{a}{4}$$

or $$w_1 a^3 = wa^4 \left(\frac{1}{3} - \frac{1}{8}\right)$$

$$= \frac{5}{24} \cdot wa^4$$

$$w_1 = \frac{5}{24} wa$$



4. Let ABCD be rectangle immersed vertically in a liquid of dnesity P, with the sides AB and CD horizontal and at depths a and b respectively. Consider an elementary strip of breadth δx at a depth x below the free surface.

δs=area of the strip
$= c \cdot \delta x$
where $AB = c$

p=pressure per unit area of the strip
$= w \cdot x.$

Let $\bar{x}$ be the depth of centre of pressure

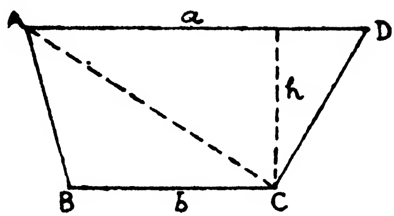
$$\bar{x} = \frac{\int xpds}{\int pds} = \frac{\int_a^b x \cdot w\, x \cdot cdx}{\int_a^b w\, x \cdot c \cdot dx}$$

$$= \frac{\int_a^b x^2\, dx}{\int_a^b xdx}$$

$$=\frac{\left[\frac{x^3}{3}\right]_a^b}{\left[\frac{x^2}{2}\right]_a^b}$$

$$=\frac{2}{3}\cdot\frac{b^3-a^3}{b^2-a^2}$$

$$=\frac{2}{3}\cdot\frac{b^2+ab+a^2}{b+a}.$$



5. Let ABCD be the trapezium. Divide the trapezium into two triangles by joining A, C.

If w be the weight of unit volume of the liquid, the thrust on the $\triangle$ ABC

$$=w\cdot\frac{bh}{2}\left(\frac{2}{3}h\right)=\frac{1}{3}wbh^2.$$

and it acts at a depth $\frac{3}{4}h$ below AD.

Thrust on the $\triangle$ ACD

$$=w\cdot\left(\frac{ah}{2}\right)\left(\frac{h}{3}\right)=\frac{1}{6}w\cdot ah^2$$

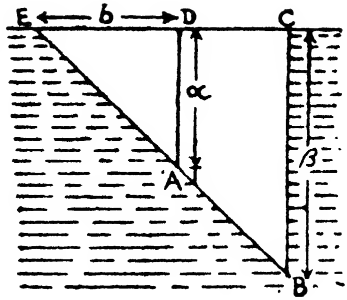
and it acts, at a depth $\frac{1}{2}h$ below AD.

Therefore, depth of the C. P. of the trapezium ABCD

$$=\frac{\frac{1}{3}w\cdot bh^2\times\frac{3}{4}\cdot h+\frac{1}{6}wah^2\times\frac{h}{2}}{\frac{1}{3}wb\cdot h^2+\frac{1}{6}wah^2}$$

$$=\frac{1}{2}h\cdot\left[\frac{b/2+a/6}{b/3+a/6}\right]$$

$$=\left(\frac{a+3b}{a+2b}\right)\cdot\frac{h}{2}.$$



6. Let BA produced meet the surface in E and let CB=β, DA=α, CE=c and DE=b.

Thrust on $\triangle$ EBC

$=w\left(\frac{1}{2}c\,.\,\beta\right)\frac{1}{3}\beta=\frac{1}{6}w\,c\beta^2$

acting at a depth $\frac{1}{2}\beta$.

Thrust on $\triangle$ EAD

$$=w\left(\tfrac{1}{2}b\alpha\right)\tfrac{1}{3}\alpha$$
$$=\tfrac{1}{6}w\,b\alpha^2$$

acting at a depth $\frac{1}{2}\alpha$.

Thrust on ABCD=thrust on $\triangle$ EBC$-$thrust on $\triangle$ EAD

$$=\tfrac{1}{6}w\,c\beta^2-\tfrac{1}{6}wb\alpha^2$$

Let $\overline{x}$ is the depth of C. P.

$$\overline{x}=\frac{\frac{1}{6}w\,c\beta^2\,.\,\frac{1}{2}\beta-\frac{1}{6}w\,b\,.\,\alpha^2\,.\,\frac{1}{2}\alpha}{\frac{1}{6}w\,c\beta^2-\frac{1}{6}wb\,.\,\alpha^2}$$

$$=\frac{1}{2}\cdot\frac{c\beta^3-b\alpha^3}{c\beta^2-b\alpha^2}\quad\text{since}\quad\frac{b}{c}=\frac{\alpha}{\beta}.$$

$$=\frac{1}{2}\cdot\frac{\alpha^4-\beta^4}{\alpha^3-\beta^3}$$

$$=\frac{1}{2}\cdot\frac{(\alpha+\beta)(\alpha^2+\beta^2)}{\alpha^2+\alpha\beta+\beta^2}.$$

7. Let ABCD be the lid, AB being the line of hinges. In each case the moment of the weight W of the lid about AB must just equal the moment of the thrust on the lid. Let the required angles be $\theta_1, \theta_2, \theta_3$ when t .e box is turned about the edges parallel to AB, CD, BC respectively. If a be the length of an edge,

$$W\,.\,\frac{a}{2}\cos\theta_1=wa^2\,.\,\frac{a}{2}\sin\theta_1\,.\,\frac{a}{3}\ldots\ldots\ldots\ldots\text{I}$$

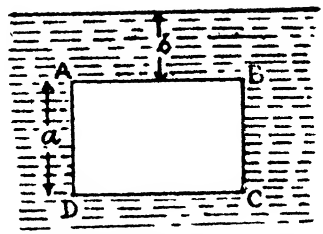
$$W\,.\,\frac{a}{2}\cos\theta_2=wa^2\,.\,\frac{a}{2}\,.\,\sin\theta_2\,.\,\frac{2a}{3}\ldots\ldots\ldots\text{II}$$

$$W \cdot \frac{a}{2} \cdot \cos\theta_3 = wa^2 \cdot \frac{a}{2} \cdot \sin\theta_3 \cdot \frac{a}{2} \ldots\ldots \text{III}$$

$$\therefore \quad \cot\theta_1 : \cot\theta_2 : \cot\theta_3 = \tfrac{1}{6} : \tfrac{2}{6} : \tfrac{1}{4}$$

$$\text{i. e.} \quad \tan\theta_1 : \tan\theta_2 : \tan\theta_3 = 6 : 3 : 4$$

EXAMPLES XXX



1. Let ABCD be the square when it is just immersed with AB in the surface, the thrust on it $\frac{a^2\,aw}{2}$ at a depth $\frac{2a}{3}$ from AB.

When it is lowered through a depth b, the addition thrust $a^2\,bw$ acts at a depth $\frac{a}{2}$ below AB.

Now if, the depth of C. P. is $\overline{x}$, below AB,

$$\overline{x} = \frac{\frac{a^3}{2} w \left(\frac{2a}{3}\right) + a^2\,bw\left(\frac{a}{2}\right)}{\frac{a^3}{2} w + a^2 bw}$$

$$= \frac{\frac{a^2}{3} + \frac{ba}{2}}{\frac{a}{2} + b}$$

Depth below the centre

$$= \frac{a\,(2a+3b)}{(3\,a+2b)} - \frac{a}{2}$$

$$= \frac{a^2}{6a+12b}.$$

2. We know that the centre of pressure of two thrusts i. e. one equal to $\frac{1}{2}\,wah \cdot \frac{2h}{3}$ acting at a depth $k-\frac{h}{4}$, and the other equal to $\frac{1}{2}\,wah\,(k-h)$ atcing at a depth $\left(k-\frac{h}{3}\right)$

Hence the depth of centre of pressure

$$=\frac{\frac{1}{2}\,wah\,.\,\frac{2h}{3}\left(k-\frac{h}{4}\right)+\frac{1}{2}\,wah\,(k-h)\left(k-\frac{h}{3}\right)}{\frac{1}{2}\,wah\,.\,\frac{2h}{3}+\frac{1}{2}\,wah\,.\,(k-h)}$$

$$=\frac{\frac{2h}{3}\left(k-\frac{h}{4}\right)+(k-h)\left(k-\frac{h}{3}\right)}{\frac{2h}{3}+(k-h)}$$

$$=\frac{\frac{2hk}{3}-\frac{h^2}{6}+k^2-\frac{kh}{3}-hk+\frac{h^2}{3}}{\frac{2h}{3}+k-h}$$

$$=\frac{4\,hk-h^2+6k^2-2kh-6hk+2h^2}{2\,(2h+3k-3h)}$$

$$=\frac{6k^2+h^2-4hk}{2\,(3k-h)}$$

3. We are required to find the resultant of two thrusts one of which is that which would act on the triangle if its upper side were in the surface of the liquid, and the other due to the superincumbent liquid of depth $k-h$.

Thrusts are

$\frac{1}{2}\,wah\,.\,\frac{h}{3}$ acting a at depth $k-\frac{h}{2}$

and the other $\frac{1}{2}\,wah\,(k-h)$ acting at a depth $k-\frac{2h}{3}$.

Hence the depth of centre of pressure

$$=\frac{\frac{1}{2}\,wah\,.\,\frac{h}{3}\left(k-\frac{h}{2}\right)+\frac{1}{2}\,wah\,(k-h)\left(k-\frac{2h}{3}\right)}{\frac{1}{2}\,wah\,(k-h)+\frac{1}{2}\,wah\,\frac{h}{3}}$$

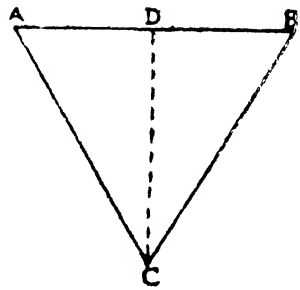
$$=\frac{\frac{h}{3}\left(k-\frac{h}{2}\right)+(k-h)\left(k-\frac{2h}{3}\right)}{\frac{h}{3}+k-h}$$

$$=\frac{\frac{hk}{3}-\frac{h^2}{6}+k^2-\frac{2hk}{3}-hk+\frac{2h^2}{3}}{\frac{h}{3}+k-h}$$

$$=\frac{2hk-h^2+6k^2-4hk-6hk+4h^2}{2\,(h+3k-3h)}$$

$$=\frac{6k^2+3h^2-8hk}{2\,(3k-2h)}.$$

4. Let the side $AB=BC=CA=6\sqrt{3}$



Area of the triangle $=\frac{1}{2}\,AB\,.\,CD$

$=\frac{1}{2}\,.\,6\sqrt{3}\times 6\sqrt{3}\sin 60$

$=3\sqrt{3}\times 3\sqrt{3}\,.\,\sqrt{3}$

$=27\sqrt{3}$

height of the triangle $=9$

We have two thrusts, one

$$w\,.\,27\sqrt{3}\,.\,\frac{9}{3}\text{ acting at a depth }\frac{9}{2}$$

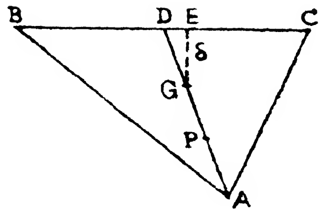
and the other $27\sqrt{3}\,w\,.\,34$ acting at a depth $\frac{9}{3}$.

$$\therefore\quad \text{Required C. P.}=\frac{(w\,27\sqrt{3})\,\frac{9}{3}\,.\,\frac{9}{2}+(27\sqrt{3}\,w)\,34\,.\,\frac{9}{3}}{(w\,.\,27\sqrt{3})\,.\,\frac{9}{3}+(27\sqrt{3}\,w)\,.\,34}$$

$$=\frac{\frac{27}{2}+102}{3+34}=\frac{231}{74}$$

$$=3\frac{9}{74}\text{ feet.}$$

5. Suppose G and P the centre of gravity and centre of pressure when the atmospheric pressure is neglected.



It is given that

$$GE=\delta$$

$$\text{Depth of } A=3\delta$$

Hence depth of P below BC is $\frac{1}{2}$ AD

$$\text{i. e.}=\frac{3\delta}{2}$$

Pressure $=\frac{1}{2}\,BC\,.\,3\delta\times\delta w=\frac{3}{2}\,BC\,.\,\delta^2 w.$

Now when the atmospheric pressure is to be considered, we shall have the following thrusts acting on the triangle.

(i) The whole pressure $\frac{3\delta^2}{2}$ BC . w acting at P i. e. at a depth $\frac{3\delta}{2}$ below BC.

(ii) The addition thrust $\frac{1}{2}\,BC\times3\delta\times hw$ due to the atmospheric pressure which will act at G i. e. at a depth 'δ' below BC.

∴ the depth of new position of C. P.

$$=\frac{\frac{3\delta^2}{2}\,BC\,.\,w\,.\,\frac{3\delta}{2}+\frac{3\delta h}{2}\,.\,BC\,w\times\delta}{\frac{3\delta^2}{2}\,BC\,w+\frac{3\delta h}{2}\,BC\,.\,w}$$

$$=\frac{\delta\left(\frac{3\delta}{2}+h\right)}{\delta+h}$$

$$=\frac{\delta\,(3\delta+2h)}{2\,(\delta+h)}$$

∴ height of new C. P. above the old C. P.

$$=\frac{3\delta}{2}-\frac{\delta\,(3\delta+2h)}{2\,(\delta+h)}$$

$$=\frac{1}{2}\cdot\frac{\delta h}{\delta+h}.$$

6. Let a be the area of triangle and k is the height.

But $\quad \delta=\frac{2k}{3}.$

We are required to find C. P. of a thrust $aw\cdot\frac{2k}{3}$ at a depth $\frac{3k}{4}$ and of a thrust $aw\cdot h$ at a depth $\frac{2k}{3}$

$$\therefore\quad \text{Required depth}=\frac{\frac{2k}{3}\cdot\frac{3k}{4}+h\cdot\frac{2k}{3}}{\frac{2k}{3}+h}$$

$$=\frac{6k^2+8hk}{4(2k+3h)}$$

$$=\frac{3k}{4}-\frac{hk}{2(2k+3h)}$$

$$=\frac{3k}{4}-\frac{h\delta}{8(\delta+h)}$$

Hence the depth$=_8\frac{h\delta}{8(\delta+h)}$.

EXAMPLES XXXI

1. Depth of C. P. below the surface$=\dfrac{\alpha^2+\beta^2+\nu^2+\alpha\beta+\beta\nu+\nu\alpha}{2\,.\,(\alpha+\beta+\nu)}$

Depth of C. G. below the surface$=\dfrac{\alpha+\beta+\nu}{3}$

$\therefore$ The C. P. is at a depth below the C. G. is given by

$$\frac{\alpha^2+\beta^2+\nu^2+\alpha\beta+\beta\nu+\nu\alpha}{2(\alpha+\beta+\nu)}-\frac{\alpha+\beta+\nu}{3}$$

$$=\frac{\alpha^2+\beta^2+\nu^2-\alpha\beta-\beta\nu-\nu\alpha}{6(\alpha+\beta+\nu)}.$$

The centre of pressure of any triangular area wholly immersed in a liquid coincides with the centre of parallel forces acting at vertices A, B, C and proportionate to $2\alpha+\beta+\nu$, $\alpha+2\beta+\nu$, $\alpha+\beta+2\nu$ respectively.

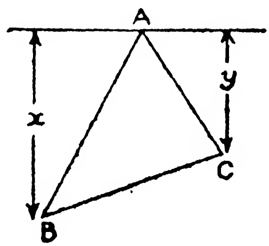
$\therefore$ The depth of the centre of the given parallel forces

$$\frac{\alpha(2\alpha+\beta+\nu)+\beta(\alpha+2\beta+\nu)+\nu(\alpha+\beta+2\nu)}{(2\alpha+\beta+\nu)+(\alpha+2\beta+\nu)+(\alpha+\beta+2\nu)}$$

$$=\frac{2\alpha^2+2\beta^2+2\nu^2+2\beta\nu+2\nu\alpha+2\alpha\beta}{4(\alpha+\beta+\nu)}$$

$$=\frac{\alpha^2+\beta^2+\nu^2+\beta\nu+\nu\alpha+\alpha\beta}{(\alpha+\beta+\nu)}.$$

2. Neglecting the atmospheric pressure the depth of C. P. is given by



$$=\frac{x^2+y^2+xy}{2(x+y)}$$

and thrust $=w\,.\,\triangle\,.\left(\frac{x+y}{3}\right)$

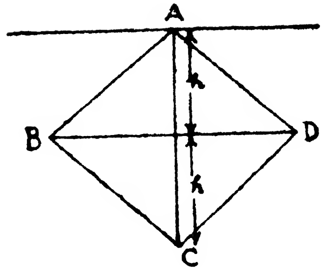
The atmospheric pressure produces an additional thrust $w\,.\,\triangle\,h$ at a depth $\frac{x+y}{3}$.

$\therefore$ Depth of C. P. of the triangle

$$=\frac{w\,.\,\triangle\,\frac{x+y}{3}\,.\left(\frac{x^2+y^2+xy}{2(x+y)}\right)+w\,.\,\triangle\,h\left(\frac{x+y}{3}\right)}{w\,.\,\triangle\left(\frac{x+y}{3}\right)+w\,.\,\triangle\,.\,h}$$

$$=\frac{\frac{x^2+y^2+xy}{6}+\frac{h(x+y)}{3}}{\frac{x+y}{3}+h}$$

$$=\frac{x^2+xy+y^2+2h(x+y)}{2(x+y+3h)}.$$



3. Let ABCD be the rhombus so that A is in the surface. Therefore depth of A, B, C, D are o, h, $2h$, h respectively.

If the depth of C. P. of ABD and BCD are z_1 and z_2 ; then

$$z_1=\tfrac{1}{2}\cdot\frac{h^2+h^2+h^2}{2h}=\frac{3}{4}\,h$$

and $$z_2=\tfrac{1}{2}\cdot\frac{h^2+h^2+4h^2+2h^2+2h^2+h^2}{h+h+2h}$$

$$=\frac{11}{8}\,h.$$

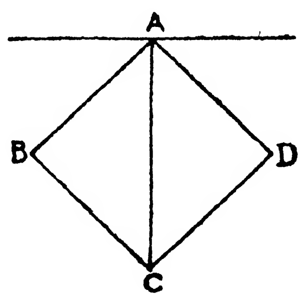
Therefore, C. P. of the rhombus

$$=\frac{w\cdot\triangle\cdot\frac{2h}{3}\cdot\frac{3}{4}\,h+w\cdot\triangle\cdot\frac{4h}{3}\cdot\frac{11}{8}h}{w\cdot\triangle\cdot\frac{2h}{3}+w\cdot\triangle\cdot\frac{4h}{3}}$$

$$=\frac{7}{6}\,h.$$

$$=\frac{7}{12}\cdot(2h)=\frac{7}{12}\cdot(AC)$$

$\therefore$ C. P. divide the vertical diagonal in the ratio 7 : 5.



4. Let ABCD be the square, A being the highest point. If AC$=d$ then the depth of A$=d$, that of C$=2d$ and that of B and D$=\frac{3d}{2}$.

The depth of the middle points of AC, CB, BA are $\frac{3d}{2}$, $\frac{7d}{4}$, and $\frac{5d}{4}$.

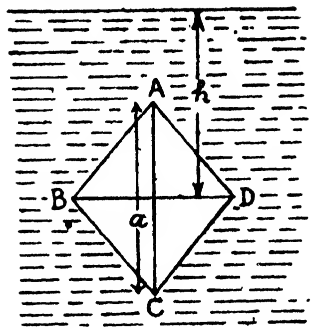
Hence the depth of C. P. of $\triangle$ ABC

$$=\frac{\left(\frac{3d}{2}\right)^2+\left(\frac{7d}{4}\right)^2+\left(\frac{5d}{4}\right)^2}{\frac{3d}{2}+\frac{7d}{2}+\frac{5d}{4}}$$

$$=\frac{55}{36}d=\frac{55}{54}\times\frac{3d}{2}$$

$$=\frac{55}{54}\times\text{depth of the centre of square.}$$

$=$which is the depth of C. P. of the square.



5. Let ABCD be the rhombus so that the depths of A, B, C, D are $h-\frac{a}{2}$, h, $h+\frac{a}{2}$, h respectively.

If the depths of the centres of pressures of ABD and BCD are z_1 and z_2, then

$$z_1=\frac{1}{2}\cdot\frac{\left(h-\frac{a}{2}\right)^2+h^2+h^2+2h\left(h-\frac{a}{2}\right)+h^2}{h+h+h-\frac{a}{2}}$$

$$=\frac{1}{4}\cdot\frac{24h^2-8ah+a^2}{6h-a}$$

Similarly $\quad z_2=\frac{1}{4}\cdot\dfrac{24h^2+8ah+a^2}{6h+a}$

Therefore the centre of pressure

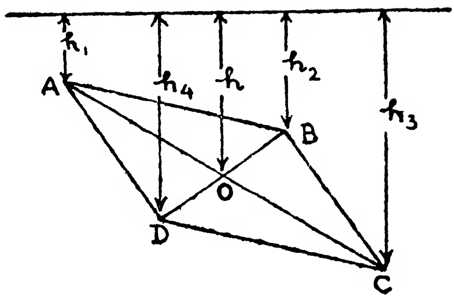
$$=\frac{w\cdot\triangle\cdot\left(h-\frac{a}{6}\right)\frac{1}{4}\cdot\frac{24h^2-8ah+a^2}{6h-a}+}{w\cdot\triangle\cdot\left(h-\frac{a}{6}\right)+}$$

$$\frac{w \cdot \Delta \cdot \left(h+\frac{a}{6}\right)\frac{1}{4} \cdot \frac{24h^2+8ah+a^2}{6h+a}}{w \cdot \Delta \cdot \left(h+\frac{a}{6}\right)}$$

$$=\frac{1}{24} \cdot \frac{48h^2+2a^2}{2h}$$

$$=h+\frac{a^2}{24h} .$$

6. Consider the parallelogram to be made of two Δs ABD and Δ BCD and find the depth of the point of application of the resultant of the parallel forces acting at the mid-points of the sides, which are proportional to their depths.



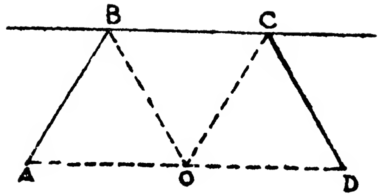
$\therefore$ Depth of C. P.

$$=\frac{k \cdot \left(\frac{h_1+h_2}{2}\right)^2 + k.\left(\frac{h_1+h_4}{2}\right)^2 + k \cdot h^2+kh^2+k \cdot \left(\frac{h_2+h_3}{2}\right)^2+k\left(\frac{h_3+h_4}{2}\right)^2}{k \cdot \left(\frac{h_1+h_2}{2}\right) + k.\left(\frac{h_1+h_4}{2}\right)+k \cdot h+k \cdot h+k \cdot \left(\frac{h_2+h_3}{2}\right)+k \cdot \left(\frac{h_3+h_4}{2}\right)}$$

$$=\frac{h_1^2+h_2^2+h_3^2+h_4^2-4h^2+(h_2+h_4)(h_1+h_3)}{2\ (h_1+h_2+h_3+h_4+2h)}$$

$$=\frac{h_1^2+h_2^2+h_3^2+h_4^2+8h^2}{12h}$$

since $h_1+h_3=2h$ and $h_2+h_4=28$.



7. Let ABCD be the half hexagon, BC being in the surface. Let $2\alpha=\frac{a\sqrt{3}}{2}$, be the depth of O, the middle point of DA.

Since the three triangles OAB, OBC and OCD, place wts. proportional to the depths at the middle points of the sides we have 6α at depth α and 4α at depth 2α.

$$\therefore \quad \text{Depth}=\frac{6\alpha\,.\,\alpha+4\alpha\,.\,2\alpha}{6\alpha+4\alpha}$$

$$=\frac{14}{10}\alpha=\frac{7}{10}\cdot\frac{a\sqrt{3}}{2}=\frac{7\sqrt{3}}{20}\,a.$$

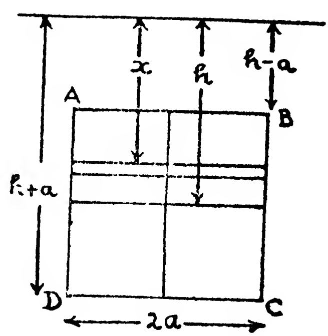
10. For the first part put $\alpha=\beta=0$ in the solution of Ex. 11. For the second part

$$3h=2\left[\frac{r+\delta}{2}-\frac{1}{6}\,\frac{r\delta}{h}\right]$$

$$\therefore \quad 9h^2-3\,(r+\delta)\,h+r\delta=0$$

Hence $h=\frac{r}{3}$ or $\frac{\delta}{3}$

i. e. Depth of C. G. of ABCD is the same either that of the $\triangle$ ACB or $\triangle$ ABD, which is the possible.



12. Let ABCD be the square. Consider an elementary strip of breadth δx at a depth x below the free surface.

If $\bar{x}$ be the depth of C. P.

$$\bar{x}=\frac{\int xp\,.\,ds}{\int p\,.\,ds}$$

$$=\frac{\int_{h-a}^{h+a} xg\,\rho\,.\,x\,2a\,dx}{\int_{h-a}^{h+a} g\rho\,.\,x\,2a\,dx}$$

$$=\int_{h-a}^{h+a} x^2\,dx \Big/ \int_{h-a}^{h+a} x\,dx.$$

$$=\frac{\frac{1}{3}\Big[x^3\Big]_{h-a}^{h+a}}{\frac{1}{2}\Big(x^2\Big)_{h-a}^{h+a}}=\frac{2}{3}\cdot\frac{[(h+a)^3-(h-a)^3]}{[(h+a)^2-(h-a)^2]}$$

$$=\frac{2}{3}\cdot\frac{[(h+a)^2+(h-a)^2+(h+a)(h-a)]}{h+a+h-a}$$

$$=\frac{2}{3}\cdot\frac{[2h^2+2a^2+h^2-a^2]}{2h}$$

$$=\frac{3h^2+a^2}{3h}$$

$\therefore$ Depth of C. P. below the centre of square

$$=\frac{3h^2+a^2}{3h}-h$$

$$=\frac{3h^2+a^2-3h^2}{3h}$$

$$=\frac{a^2}{3h}.$$

EXAMPLES XXXII

1. From Ex. I, we get

$$\pi\rho r^2\left(gh+\frac{w^2r^2}{4}\right)=\frac{3}{2}\times\pi\rho r^2(gh+r)$$

or
$$\frac{w^2r^2}{4}=\frac{3}{2}gh-gh$$
$$=\tfrac{1}{2}gh$$

i. e. $w=\sqrt{\dfrac{2\,gh}{r^2}}$ where w is the angular velocity.

2. From the previous question, we get

$$\pi\rho r^2\left(gh+\frac{w^2r^2}{4}\right)=5\times\tfrac{1}{4}\pi\rho r^4w^2$$

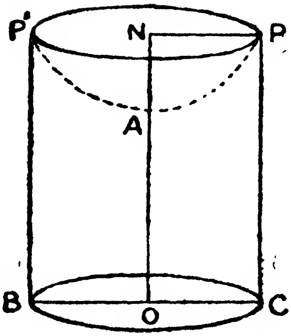
$$gh+\frac{w^2r^2}{4}=\tfrac{5}{4}r^2w^2$$

i. e. $w=\dfrac{\sqrt{gh}}{r}.$

3. From the previous question,

$$gh + \frac{w^2 r^2}{4} = \frac{n}{4} \times r^2 w^2$$

$$w = \frac{2}{r} \sqrt{\frac{gh}{n-1}}.$$



4. Pressure at a point of the base distant y from O is given by

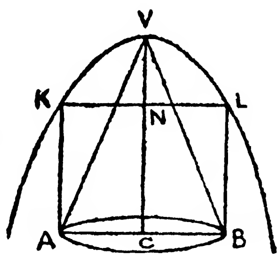
$$= \rho \left[\tfrac{1}{2} w^2 . y^2 + g . AO\right]$$

But $\quad AO = h - AN = h - \dfrac{w^2 r^2}{2g}$

Hence pressure $= \rho\left[\dfrac{1}{2} w^2 y^2 + gh - \dfrac{w^2 r^2}{2}\right]$

$$= g\rho \left[h - \frac{w^2}{2g}\left(r^2 - y^2\right)\right]$$

at a distance y from the centre.



5. Let a section through the vertex V of the cone cut the base in the line ACB. Draw a parabola throngh V, with the axis vertical and let. the latus rectum is $\dfrac{2g}{w^2}$. Let KL meets the axis in N.

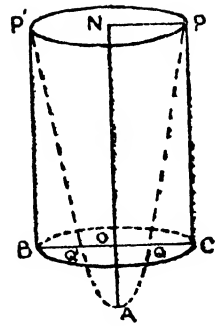
$$a^2 = NL^2 = \frac{2g}{w^2} . NV$$

$$\therefore \qquad VN = \frac{w^2 a^2}{2g}.$$

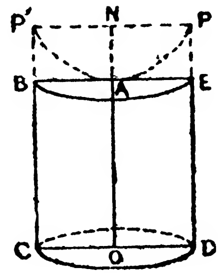
Therefore pressure on base = weight of volume ABLVK

$$6 \pi a^2 . \tfrac{1}{3} a \cot \alpha = \pi a^2 (a \cot \alpha + VN) - \tfrac{1}{2} \pi a^2 . VN$$

$$\therefore \qquad w = \sqrt{\frac{4g \cot \alpha}{a}}.$$



6. If $\frac{w^2 r^2}{2g}=h$, then NA=NO ; and the vertex A of the parabola coincides with the lowest point of the axis i. e. when $w=\frac{\sqrt{2gh}}{r}$ and therefore the volume of the paraboloid $PAP'=\frac{1}{2}$ that of the cylinder. Hence in this case the water left in the cylinder is of volume half that of cylinder.



7. The lid will turn round the hinge B if its weight

=the weight of water of volume P'APEAB

$=\frac{1}{2}\pi a^2 . AN . g\rho$

$=\frac{\pi a^2 g\rho}{2}\times\frac{w^2 a^2}{2g}$

$=\frac{1}{4}\pi a^4 w^2\rho.$

8. From question 6, we get $N=\frac{\sqrt{2gh}}{r}$

Volume left$=\frac{1}{n}$ *th* original volume.

$$=\frac{1}{n}\times\frac{\pi}{2} r^2 h$$

Hence $\frac{\pi}{2n} r^2 h=\pi r^2 h - \frac{\pi h}{w^2}(w^2 r^2-gh)$

$$=\frac{\pi g h^2}{w^2}$$

and therefore $w^2=\frac{\sqrt{2ngh}}{r}=\sqrt{n} . N.$

9. Let us suppose the centre of tube is O, this tube revolve round the tangent at C. Let the parabolic free surface cut tangent at C in A and the circle in P, Q, so that

e two latter points are at the end of a diameter. Let θ is e angle that PQ makes with CO produced. Draw PN, QM rp. to CA. Hence

$$PN = a + a\cos\theta$$
$$QM = a - a\cos\theta$$

Therefore,

$$(a + a\cos\theta)^2 = \frac{2g}{w^2} \cdot AN$$

Also $$(a - a\cos\theta)^2 = \frac{2g}{w^2} \cdot AM$$

Subtracting, we get,

$$4a^2\cos\theta = \frac{2g}{w^2}(AM - AN)$$
$$= \frac{2g}{w^2} \cdot MN$$
$$= \frac{2g}{w^2} \cdot 2a\sin\theta.$$
$$\tan\theta = \frac{w^2 a}{g}$$
$$\theta = \tan^{-1}\frac{w^2 a}{g}.$$

10. Let us assume that the parabolic free surface meet he tube in P and Q, and the vertical through the centre C n A. Draw PN, QM perp. to CA.

$$\therefore \quad \angle PCN = 60° \quad \text{and} \quad \angle QCM = 30°.$$

so that $$PN = \frac{a\sqrt{3}}{2} \quad \text{and} \quad QM = \frac{a}{2}$$

$$\therefore \quad \frac{3a^2}{4} = PN^2 = \frac{2g}{w^2} \cdot AN$$
$$\frac{a^2}{4} = QM^2 = \frac{2g}{w^2} \cdot AM.$$

Subtracting, we get

$$\frac{3a^2}{4}-\frac{a^2}{4}=\frac{2g}{w^2}\,.\,\text{MN}$$

$$\frac{a^2}{2}=\frac{2g}{w^2}\left(\frac{a\sqrt{3}}{2}+\frac{a}{2}\right)$$

$$w^2=\frac{2g}{a}\,(\sqrt{3}+1).$$

11. If C be the centre of the tube, A its lowest point. The parabolic free surface will have its vertex at A and pass through a point P on the tube such that $\angle$ACP$=\theta/2$. Draw PN perp. to AC. Hence,

$$a^2\sin^2\frac{\theta}{2}=\text{PN}^2=\frac{2g}{w^2}\,.\,\text{AN}$$

$$=\frac{2g}{w^2}\left(a-a\cos\frac{\theta}{2}\right)$$

$$=\frac{2g}{w^2}\,.\,2a\,.\,\sin^2\frac{\theta}{4}$$

$$w^2=\frac{g}{a}\;\frac{4\sin^2\dfrac{\theta}{4}}{\sin^2\dfrac{\theta}{2}}=\frac{g}{a}\;\frac{1}{\cos^2\dfrac{\theta}{4}}$$

$$w=\sqrt{\frac{g}{a}}\sec\frac{\theta}{4}.$$

12. Let the vertical and horizontal radii of the tube be CB and CP. Draw a parabola, latus rectum $\frac{2g}{w^2}$ to go through P and cut the tube in Q and CB in A. Draw QN perpendicular to CA. Then, from the question

$$\angle\text{BCQ}=45^\circ\quad\therefore\quad\text{QN}=\frac{a}{\sqrt{2}}$$

Therefore,

$$a^2=\text{CP}^2=\frac{2g}{w^2}\,.\,\text{AC}.$$

Also $$\frac{a^2}{2}=\frac{2g}{w^2}\,.\,\text{AN}$$

$\therefore$ $$\frac{a^2}{2}=\frac{2g}{w^2}\,.\,\text{CN}$$

$$=\frac{2g}{w^2}\cdot\frac{a}{\sqrt{2}}$$

$$\therefore\quad w^2=\frac{2g}{a}\cdot\sqrt{2}$$

Also $$\text{CA}=\frac{w^2\,a^2}{2g}=a\sqrt{2}.$$

13. From question No. 12, $\angle \text{QCN}=60^\circ$.

Therefore

$$a^2=\text{CP}^2=\frac{2g}{w^2}\,.\,\text{AC}$$

Also $$\frac{a^2\,.\,3}{4}=\text{QN}^2=\frac{2g}{w^2}\,.\,\text{AN}$$

$\therefore$ $$\frac{a^2}{4}=\frac{2g}{w^2}\,.\,\text{CN}$$

$$=\frac{2g}{w^2}\,.\,a\,.\,\frac{1}{2}$$

$$\therefore\quad w=2\sqrt{\frac{g}{a}}\,.$$

14. If BC, CD, DE be the sides of the tube, O the middle point of the horizontal side CD. Through E, B draw free parabolic surface to meet CD in P and Q and the vertical through O in A. Let EB meet this vertical in N.

$$\left(\frac{a}{2}\right)^2=\text{EN}^2=\frac{2g}{w^2}\,.\,\text{AN}$$

$$\therefore\quad \text{AO}=\frac{w^2\,a^2}{8g}-a$$

and $$\text{BP}^2=\frac{2g}{w^2}\,.\,\text{AO}=\frac{a^2}{4}-\frac{2ga}{w^2}$$

$\therefore$ Reqd. length$=2\,.\,\text{OP}=a\sqrt{1-\dfrac{8g}{w^2a}}$

This value is imaginary i. e. nothing escapes,

$$\text{if } w > \sqrt{\frac{8g}{a}}$$

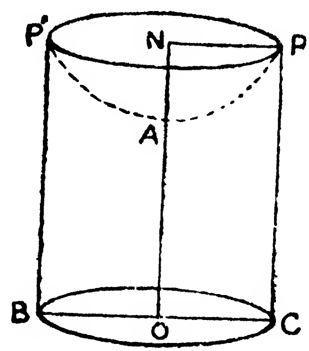
16. If a vertical plane through the axis cut the cylinder in the lines PC, CB, BP′ and let O be the middle point of BC ; and let PP′ meet the axis ON in N.

Let the free surface through P, P′ meet axis in A. If $b > \frac{1}{2}h$, i. e. if the quantity of the liquid > half the cylinder, then A is above O.

$$\pi a^2(h-b)=\pi a^2\,.\,\tfrac{1}{2}\,\text{AN}$$

$$\therefore \quad h-b=\frac{1}{2}\,.\,\text{AN}=\frac{1}{2}\,\frac{w^2}{2g}\,.\,\text{PN}^2=\frac{w^2a^2}{4g}$$

$$\therefore \quad w=\frac{2}{a}\sqrt{g\ (h-b)}$$



If $b < \frac{1}{2}h$, A is below O.

Also

$$\pi a^2(h-b)=\frac{\pi h}{w^2}(w^2a^2-gh)$$

$$w=\frac{h}{a}\sqrt{\frac{g}{b}}$$

17. We are given

$$\left(\frac{a}{2}\right)^2=\frac{2g}{w^2}\,.\,a$$

$$\therefore \quad w=\sqrt{\frac{8g}{a}}.$$

18. Let a vertical section of the cone through the vertex V cut the base in PP′, the axis of the cone meeting the parabolic free surface in A and PP′ in N.

Hence

$$h^2 \tan^2 \alpha = PN^2 = \frac{2g}{w^2} . AN$$

It is given paraboloid PAP′=half the cone

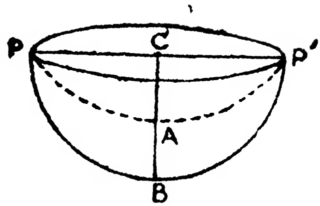
$$\text{i. e.} \quad \pi a^2 . \tfrac{1}{2} AN = \tfrac{1}{2} \pi a^2 . \tfrac{1}{3} VN$$

$$\therefore \quad \frac{w^2 h^2 \tan^2 \alpha}{2g} = \frac{1}{3} h$$

$$\therefore \quad w^2 = \frac{2g}{3h} \cot^2 \alpha$$

if $w > \sqrt{\dfrac{2g}{3h}} . \cot \alpha$, water overflows.

19. Let the plane of the paper cut the base of the hemisphere in P and P′. Draw a parabola through P and P′, of latus rectum $\dfrac{2g}{w^2}$.



Then $a^2 = CP^2 = \dfrac{2g}{w^2} CA$

$$\therefore \quad CA = \frac{w^2 a^2}{2g}$$

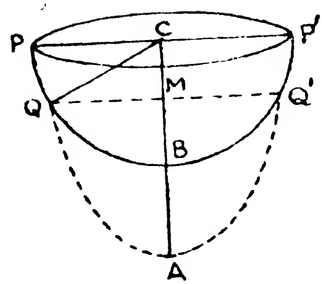
If $w^2 a < 2g$, then A is above the lowest point B of the sphere, and the amount that has run over

=Volume of paraboloid PAP′

$$= \pi a^2 . \frac{1}{2} CA = \frac{1}{4} \frac{\pi w^2 a^4}{g}.$$

If $w^2 a > 2g$, then A is below B and the parabola meets the sphere again in points Q and Q′. Let QQ′ cuts CB in M, therefore

$$a^2 - CM^2 = QM^2 = \frac{2g}{w^2} . AM$$



$$=\frac{2g}{w^2}\,.\,(CA-CM)$$

$$=a^2-\frac{2g}{w^2}\,.\,CM$$

$$\therefore \quad CM=\frac{2g}{w^2}=x.$$

Hence the amount that has run over

$$=\pi a^2\,.\,\frac{1}{2}\,AC-\pi\,QM^2\,.\,\frac{1}{2}\,AM+\frac{\pi}{3}\,(2a^3-3a^2x+x^3)$$

$$=\frac{\pi}{6}\,(4a^3-3a^2x+3x^2\,.\,AC-x^3)$$

$$=\frac{\pi}{6}\,(4a^3-x^3)$$

$$=\frac{2\pi}{3}\left(a^3-\frac{2g^3}{w^6}\right)$$

20. From question No. 18, we get

$$h^2\tan^2\alpha=PN^2=\frac{2g}{w^2}\,.\,AN$$

Amount that flows over $=\pi h^2\tan^2\alpha\,.\,\frac{1}{2}\,AN$

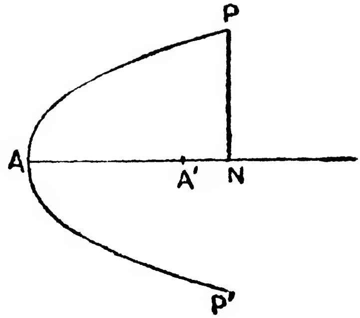
$$=\frac{1}{4}\,.\,\frac{\pi\,w^2\,h^4\,.\,\tan^4\alpha}{g}$$

In the limiting case VP is a tangent to the parabola

$\therefore$ VA=AN ; Hence

$$h=2\,AN=\frac{w^2\,h^2\tan^2\alpha}{g}$$

$$\therefore \quad w=\sqrt{\frac{g}{h}}\,.\,\cot\alpha.$$



21. Let the cup be formed by revolution of the parabola PAP' about its axis AN. Draw PN perp. to the axis and through P, P' draw the free surface viz. a parabola of latus rectum $\frac{2g}{w^2}$. Let the vertex be A' in AN.

Then $\qquad 4ah = PN^2 = \frac{2g}{w^2} . A'N$

$\therefore$ Paraboloid PAP'$-$paraboloid PA'P'
$\qquad\qquad = $Total quantity of liquid

i. e. $\quad \pi . 4ah . \frac{1}{2} h - \pi . 4ah . \frac{1}{2} A'N = \pi . 4a . \frac{h}{2} . \frac{1}{2} . \frac{1}{2} h.$

$$\therefore \qquad A'N = \frac{3h}{4}$$

From I we get $4a = \frac{2g}{w^2} . \frac{3}{4}$

$$\therefore \qquad w = \frac{1}{4}\sqrt{\frac{6g}{2}}.$$

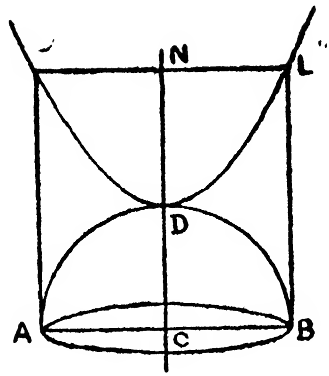
22. Let the cup be formed by revolving of the parabolic arc PCQ about the axis CN. Draw parabolic free surface through P and Q, latus rectum $\frac{2g}{w^2}$, with its vertex A on CN. If A above C, there is always water at C until the whole amount has gone out through a hole at C. If A be above C, and λ be the latus rectum of the parabola of the cup, then

$$\lambda . NC = PN^2 = \frac{2g}{w^2} . AN$$

$$\therefore \quad \lambda = \frac{2g}{w^2} . \frac{AN}{NC}$$

$$\therefore \quad \lambda < \frac{2g}{w^2}.$$

23. Let the bowl be formed by the rotation of the semicircle about the vertical axis.



Through D, draw a parabola with its axis vertical and latus rectum $\frac{2g}{w^2}$; and let the vertical lines through A, B meet it in K and L respectively.

Since the bowl is on the point of rising, therefore, its weight =upward psessure on the bowl

=wt. of AKDLBDA

=wt. of $\left(\text{cylinder ABLK} - \frac{1}{2}\text{sphere ADB} - \text{paraboloid KDL}\right)$

$$= w \cdot \pi a^2 \left(AK - \frac{2}{3}a - \frac{1}{2}DN\right)$$

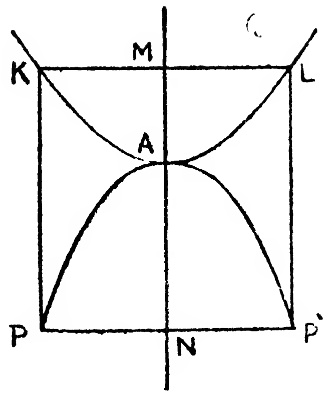
Now $\quad a^2 = \frac{2g}{w^2} \cdot ND$

$$\therefore \quad \frac{\text{weight of hemisphere}}{\text{weight of contained water}} = \frac{AK - \frac{2}{3}a - \frac{1}{2}DN}{\frac{2}{3}a}$$

$$= \frac{\frac{a}{2} + \frac{1}{2}DN}{\frac{2}{3}a}$$

$$= \frac{1}{2} + \frac{3}{4} \cdot \frac{w^2 a}{2g}$$

$$= \frac{4g + 3w^2 a}{8g}.$$



24. Let the cup be formed by the revolution of the parabola PAP′ about its axis. Draw a parabola, A as vertex and $\frac{2g}{w^2}$ as latus rectum

Cup on the point of rising when

W=upward pressure of liquid

W=wt. [cylinder PKLP′−paraboloid PAP′−paraboloid KAL]

$=w_1 . \pi . PN^2 (NM-\frac{1}{2} AN-\frac{1}{2} AM)$

$=w_1 . \pi . PN^2 . \frac{1}{2} (AN+AM)$

Let AN$=h$ $\therefore$ $PN^2=4ah$

$$AM=\frac{w^2}{2g} . MK^2=\frac{w^2}{2g} . 4ah$$

$$\therefore \quad W=w_1 . \pi . 2ah\left(h+\frac{w^2}{g} . 2ah\right)$$

Also $\quad W=w_1 . \pi . 4ah . \frac{1}{2} h$

$$\therefore \quad \frac{W}{w}=1+\frac{w^2}{g} . 2a$$

i. e. $\quad w=\sqrt{\frac{W-w}{w} . \frac{g}{2a}}$

26. Let the section of the cone by a plane through its axis VAN be VP and VP′. Through P, P′ draw a parabola of latus rectum $\frac{2g}{w^2}$, having its vertex A on VN.

Also $\quad \frac{h^2}{3}=h^2 \tan^2 30=PN^2$

$$=\frac{2g}{w^2} . AN$$

$$\therefore \quad AN=\frac{w^2 h^2}{6g}.$$

For the equilibrium, we get

$$\rho \cdot \tfrac{1}{3}\pi\, PN^2 \cdot h = \text{upward pressure of the liquid}$$

$$= \frac{4\rho}{3}\ (\text{volume of cone—volume of PAP}')$$

$$\therefore\quad \tfrac{1}{3}h = \frac{4}{3}\left(\frac{1}{3}h - \frac{1}{2}AN\right)$$

$$\frac{h}{3} = 2\,AN = \frac{w^2 h^2}{3g}$$

$$\therefore\quad w = \sqrt{\frac{g}{h}}\,.$$

28. Let the string make an angle θ to the vertical. Then we have

$$mg - T\cos\theta = \frac{mg}{\sigma} \ldots\ldots\ldots\ldots \text{I}$$

and $$mw^2y = \frac{m}{\sigma}w^2y + T\sin\theta$$

$$\therefore\quad mw^2l = \frac{mw^2l}{\sigma} + T \ldots\ldots\ldots\ldots\ldots \text{II}$$

From I and II

$$mg\left(1-\frac{1}{\sigma}\right) = \cos\theta \cdot mw^2l\left(1-\frac{1}{\sigma}\right)$$

$$\therefore\quad \cos\theta = \frac{g}{w^2l} \ldots\ldots\ldots\ldots\ldots \text{III}$$

This will give real values for θ; if

$$w^2l > g$$

$$\text{i. e.}\qquad w > \sqrt{\frac{g}{l}}\,.$$

EXAMPLES XXXIII

1. Let x be the immersed depth

$$x \cdot \sigma = \rho \cdot 2b$$

$$HM = \frac{\lambda \cdot 2a \cdot \frac{a^2}{3}}{\lambda \cdot x \cdot 2a} = \frac{a^2}{6b} \cdot \frac{\sigma}{\rho}.$$

The equilibrium is stable if

$$\text{HM} > b-\frac{x}{2}$$

i. e.
$$\frac{a^2}{6b}\cdot\frac{\sigma}{\rho} > b-\frac{\rho}{\sigma}\,.\,b.$$

$$\frac{a^2}{6b^2} > \frac{\rho}{\sigma}-\frac{\rho^2}{\sigma^2}.$$

2. If the immersed depth$=\frac{c}{2}$

Therefore
$$\text{HM}=\frac{ab\,.\,\frac{b^2}{12}}{ab\,.\,\frac{c}{2}}=\frac{1}{6}\cdot\frac{b^2}{c}$$

Equilibrium is stable if

$$\text{HM} > \frac{c}{2}-\frac{c}{4}$$

$$> \frac{c}{4}$$

$$b^2 > \frac{3}{2}c^2$$

$$b > \frac{1}{4}\sqrt{6}c.$$

The condition of stability for a rotation about the axis parallel to b, $a > \frac{1}{4}\sqrt{6}c$, is clearly satisfied

since $a > b$.

3. Let $2l$ be the length and a the radius of the base of the cylinder.

Therefore

$$\text{HM}=\frac{4\,al\,.\,\frac{l^2}{3}}{\frac{1}{2}\pi a^2\,.\,2l}=\frac{4}{3\pi}\cdot\frac{l^2}{a}$$

Distance of C. G. of a semi-circle from the centre is given by $\frac{4a}{3\pi}$.

Equilibrium is stable if

$$\frac{l^2}{a} > a$$

$$\text{i. e.} \quad l > a$$

i. e. height > diameter.

4. Let x be the length of the axis immersed

Therefore,

$$\frac{b}{h-x}=\frac{a}{h}\ldots\ldots\ldots\ldots\text{I}$$

and $$\left[\frac{1}{3}\pi a^2 h-\frac{1}{3}\pi b^2(h-x)\right]\sigma=\frac{1}{3}\pi a^3 h\rho$$

$$\therefore \quad [h^3-(h-x)^3]\,\sigma=h^3\rho\ldots\ldots\ldots\text{II}$$

$$\text{HM}=\frac{\pi b^2 \cdot \frac{b^2}{4}}{\frac{1}{3}\pi a^2 h-\frac{1}{3}\text{A}b^2(h-x)}$$

$$=\frac{3}{4}\cdot\frac{b^4}{a^2h-b^2(h-x)}$$

If V be the vertex

$$\frac{1}{3}\pi b^2(h-x)\cdot\frac{3}{4}(h-x)+\left[\frac{1}{3}\pi a^2h-\frac{1}{3}\pi b^2(h-x)\right]\text{VH}$$
$$=\frac{1}{3}\pi a^2 h\cdot\frac{3}{4}h$$

$$\therefore \quad \text{VH}=\frac{3}{4}\times\frac{a^2h^2-b^2(h-x)^2}{a^2h^2-b^2(h-x)}$$

$$\therefore \quad \text{HG}=\frac{3}{4}\cdot\frac{a^2h^2-b^2(h-x)^2}{a^2h^2-b^2(h-x)}-\frac{3}{4}h$$

$$=\frac{3}{4}\cdot\frac{xb^2(h-x)}{a^2h-b^2(h-x)}$$

Equilibrium is stable if

$$\text{HM} > \text{HG}$$

$$b^2 > x(h-x)$$
$$a^2(h-x) > h^2x \text{ from I}$$
$$a^2 h > (h^2+a^2)\,x$$
$$x < h\sin^2\alpha$$
$$\therefore \quad h-x=h\cos^2\alpha$$

From II

$$\frac{\rho}{\sigma} < \frac{1}{h^3}\left[h^3-h^3\cos^6\alpha\right]$$
$$\frac{\rho}{\sigma} < 1-\cos^6\alpha$$

5. The movement produces the same result as the placing of $-m$ tons on one side of the ship and of $+m$ tons on the other. The moment of the altered weight of the ship about G

$$=\text{moment of this couple}=mlg$$

This is balanced by the moment of the buoyancy acting at the new centre of buoyancy H′, and this moment

$$=Mg\times GM\sin M'\,Mg$$
$$\therefore \quad ml=Mh\sin M'\,MG$$
$$\therefore \quad \angle\, M'MG=\sin M'\,MG=\frac{ml}{Mh}$$

because $\angle$ M′MG is very small.

6. We are given

$$M=9000,\quad m=20,\quad l=42$$
$$\therefore \quad \frac{10}{12}=20\times\angle\, M'MG$$

From the previous question

$$\frac{10}{240}=\frac{20}{9000}\times\frac{42}{h}$$
$$\therefore \quad h=\frac{24\times 20\times 42}{9000}$$
$$=\frac{16\times 14}{100}$$
$$h=2{\cdot}24 \text{ feet.}$$

EXAMPLES XXXIV

1. Let r, k be the radius and thickness of the second, and $3r$, $2k$ that of the first.

If t be the common tensil strength,

$$P_1 = \frac{2 \cdot t \cdot 2k}{3r}$$

And $$P_2 = \frac{2\, t \cdot k}{r}$$

$$\therefore \quad \frac{P_2}{P_1} = \frac{3}{2}$$

2. From the usual notation,

$$t = 12000, \quad r = 6$$

$$\therefore \quad P = \frac{t \cdot k}{r} = \frac{12000 \times \frac{1}{4}}{6}$$

$$= 500 \text{ lbs. per sq. inch.}$$

3. $P = 200 \times 62{\cdot}5$ lb. per sq. feet.

$$= 200 \times 62{\cdot}5 \times \frac{1}{144} \text{ lbs. per sq. inch.}$$

Hence $$200 \times 62{\cdot}5 \times \frac{1}{144} = \frac{10000 \times k}{4}$$

where k is the thickness.

$$\therefore \quad k = \frac{5}{144} \text{ inch.}$$

4. With the usual notation

$$P_o = \frac{2t_o}{a} \text{ and } P_1 = \frac{2t_1}{r}$$

$$\therefore \quad \frac{P_1}{P_o} = \frac{a}{r} \cdot \frac{t_1}{t_o}$$

$$= \frac{a}{r} \cdot \frac{r^2}{a^2}$$

$$= \frac{r}{a}.$$

Also, from the formula

$$P_o \cdot a^3 = \frac{P_1 r^3}{1+\alpha t}$$

$$\therefore \quad 1+\alpha \cdot t = \frac{P_1 r^3}{P_o \cdot a^3}$$

$$= \frac{r^4}{a^4}$$

$$r = a\,(1+\alpha \cdot t)^{1/4}$$

MISCELLANEOUS EXAMPLES

1. Let x be the required depth.

Then from the condition of floating bodies

$$x \times 1{\cdot}025 = 100 \times {\cdot}92$$

$$x = 89\frac{31}{41} \text{ yards.}$$

2. Let the density of wood be ρ and volume V.

Hence

$$V \cdot \rho = 1000$$

Also $$\rho \cdot 1 = 1 \cdot \frac{3}{5}$$

$$\therefore \quad \rho = \tfrac{3}{5} = {\cdot}6$$

Therefore $$V = 1000 \times \frac{5}{3} = \frac{5000}{3}$$

$$= 1666\frac{2}{3} \text{ cub. cm.}$$

3. Let n be the required fraction of volume immersed. Therefore,

$$n \times {\cdot}84 = 1 \times \frac{2}{3}$$

$$\therefore \quad n = \frac{50}{63}.$$

4. The stopper must have the weight

=thrust due to pressure (772—730) m.m. of mercury on its face.

$$=\pi\left(\frac{5}{2}\right)^2 \cdot \frac{42}{10} \text{ cub. cm. wt.}$$
$$=357\ \pi \text{ grammes.}$$

5. If V and V′ be the volumes of gold and silver in the crown, also let ρ and ρ' be the sp. gr. of gold and silver. Therefore,

$$V \cdot 1+V' \cdot 1=\frac{1}{14}\left(V \cdot \rho+V' \cdot \rho'\right)\text{......I}$$

$$V \cdot 1=\frac{4}{77} \cdot V \cdot \rho\text{......................II}$$

$$V' \cdot 1=\frac{2}{21} \cdot V' \cdot \rho'\text{..................III}$$

From II, $\rho=\dfrac{77}{4}$

From III, $\rho'=\dfrac{21}{2}$

Reqd. proportion in weights$=\dfrac{V \cdot \rho}{V' \cdot \rho'}$,

$$=\frac{\rho}{\rho'} \cdot \frac{14-\rho'}{\rho-14}=\frac{11}{9}$$
$$=\frac{11}{9}.$$

6. Let the external side of the cube be x inches and the vessel just floats. Therefore, from the usual formula,

$$[x^3-(x-2)^3]\times 2\frac{34}{91}=x^3\times 1$$
$$\therefore \quad 125x^3=216\ (x-2)^3$$
$$\therefore \quad x=12$$
$$\therefore \quad (x-2)^3=(12-2)^3$$
$$=10^3=1000$$

which is the internal volume

7. From the Principle of Archimedes,

$$\left(\frac{2}{3}\pi r^3+\frac{1}{3}\pi r^2 \, . \, 2r\right)\rho_1=\frac{2}{3}\pi r^3 \, . \, \rho_2$$

$$\frac{4}{3}\pi r^3 \, . \, \rho_1=\frac{2}{3}\pi r^3 \, . \, \rho_2$$

$$\therefore \quad \rho_2=2\rho_1.$$

8. Let A be the cross-sectional area, h the height and be the density of the cylinder.

Therefore, from the condition of floating bodies

$$A \, . \, h \, . \, \sigma=2=A\,(h-7) \, . \, 1$$

Also $$A \, . \, h \, . \, \sigma+\frac{1}{2}=A\,h \, . \, 1$$

$$\therefore \quad A=\frac{1}{14},\ h=35, \text{ and } \sigma=\frac{4}{5}$$

x be the length above the surface,

$$A\,(h-x) \, . \, 5=A \, . \, h \, . \, \sigma$$

$$(h-x)=\frac{1}{5} \, . \, 35 \, . \, \frac{4}{5}$$

$$=5{\cdot}6$$

$$\therefore \quad x=35-5\,6$$

$$=29{\cdot}4.$$

9. Let the required gas be V cubic feet.

Therefore, upward thrust

$$=\left(V\times1\frac{1}{4}-V\times1\frac{1}{4}\times{\cdot}52\right)$$

$$=\frac{3V}{80}\text{ lbs. wt.}$$

Also, $$\frac{3V}{80}=1200$$

$$\therefore \quad V=32000.$$

10. Let W be the required weight to sink.

Hence

$$W+8=\pi\left(\frac{3}{2}\right)^2\times\frac{9}{2}\times\frac{1000}{1728}$$

$$W=\frac{375\,\pi}{64}-8$$

$$=\text{about } 10{\cdot}4 \text{ ozs.}$$

11. Let the specific gravity be ρ and x inch be the immersed length. Therefore,

$$4\times\rho+8\times{\cdot}24=x\times1\ldots\ldots\ldots\ldots\text{I}$$

Since the rod is floating, therefore sum of the moments about one end must be zero.

$$\therefore\quad 4\times2\times\rho+8\times8\times{\cdot}24=x\times\frac{x}{2}\times1\ldots\ldots\ldots\ldots\text{II}$$

Therefore, $\rho=1{\cdot}32$ and $x=7{\cdot}2$.

12. We can assume the volume of wood and lead be V and V′, and σ, σ' be the specific gravities of the two. Therefore,

$$V\sigma w=6\ldots\ldots\ldots\ldots\ldots\ldots\text{I}$$

$$V'(\sigma'-1)\,w=12\ldots\ldots\text{II}$$

$$V(\sigma-1)\,w+V'(\sigma'-1)\,w=10\ldots\ldots\text{III}$$

From II and III, we get

$$V(\sigma-1)\,w=-2$$

Therefore,

$$\frac{\sigma-1}{\sigma}=-\frac{2}{6}=-\frac{1}{3}$$

$$\therefore\quad \sigma=\frac{3}{4}={\cdot}75.$$

13. Since the body floats half immersed, the sp. gr. of the first liquid is 2ρ.

The density of the mixture $=\frac{1}{2}(2\rho+1)=\rho+\frac{1}{2}$.

Therefore

$$\left(\rho+\frac{1}{2}\right)\times\frac{3}{4}=\rho\times 1$$

$$\therefore \quad \rho=1{\cdot}5.$$

14. The pressure on a certain day is that due to

$=76$ cm. of mercury $+3{\cdot}4$ of water

$=76\times 13{\cdot}6+3{\cdot}4$ of water.

Second day the head similarly

$$=74\times 13{\cdot}6+3{\cdot}4$$

If V be the volume to which with this pressure 3 litres has increased. Therefore

$$\frac{V}{3}=\frac{76\times 13{\cdot}6+3{\cdot}4}{74\times 13{\cdot}6+3{\cdot}4}$$

$$V=3\frac{8}{99} \text{ litres.}$$

$\therefore$ Reqd. fraction of air that has bubbled out

$$=\frac{8/99}{3\frac{8}{99}}=\frac{8}{305}.$$

15. Let us assume that the surface of the mercury inside the tube rise y inches, and that the upper end of the water move x inches.

Area of the section of the bulb $=6^2=36$ i. e. 36 times the area of the tube.

But the volume of water is constant

$$\therefore \quad x=36\,y$$

If $\sigma=$ sp. gr. of mercury

Pressure increased $=\frac{1}{2}\,\sigma=y\sigma+(x-y)\,.\,1$

$$\therefore \quad \frac{1}{2}\,\sigma=\frac{x\sigma}{36}+\left(x-\frac{x}{36}\right)$$

$$18\sigma=x\sigma+35x$$

$$\therefore \quad x=\frac{18\,\sigma}{\sigma+35}=\frac{18\times 13{\cdot}67}{13{\cdot}67+35}$$

$$=5\ 05 \text{ inches.}$$

16. Required Ratio

$$=\frac{\text{difference of weights of air and gas in II case}}{\text{difference of weights of air and gas in I case}}$$

$$=\frac{750-\frac{1}{10}\cdot 760}{760-\frac{1}{10}\cdot 760}=\frac{750-76}{760-76}=\frac{674}{684}$$

$$=\frac{337}{342}.$$

17. When the sphere is in equilibrium, suppose II′, II be the pressures of air below and above the sphere. Hence from the condition

difference of pressure$=\pi r^2\, kw=\pi r^2\,(\Pi'-\Pi)$

Therefore, $\quad kw=\Pi'-\Pi$.........I

From Boyle's law

$$\Pi'\,[\pi r^2\,(h-x)-\tfrac{2}{3}\,\pi r^3]=\Pi\,(\pi r^2 h-\tfrac{2}{3}\,\pi r^3]$$

$$\therefore\quad \Pi'\left[h-x-\frac{2}{3}\,r\right]=\Pi\left(h-\frac{2r}{3}\right)\text{.........II}$$

From I, $\Pi'=kw+\Pi$; substitute in II

$$\left(kw+\Pi\right)\left(h-x-\frac{2r}{3}\right)=\Pi\left(h-\frac{2r}{3}\right)$$

$$\therefore\quad kw\left(h-x-\frac{2r}{3}\right)=\Pi\,.\,x=w\,.\,H\,.\,x$$

$$\therefore\quad x\left(H+k\right)=k\left(h-\frac{2r}{3}\right).$$

18. When the liquids are first mixed, let ρ_1, σ_1 be the sp. gravities and ρ_2, σ_2 after the second, and so on, then from the usual formula

$$\rho_1=\frac{n-1}{n}\,\rho+\frac{\sigma}{n}\text{ ....I}$$

$$\text{and}\qquad \sigma_1=\frac{\rho}{n}+\frac{n-1}{n}\,\sigma\text{.........II}$$

$$\therefore \quad \rho_1-\sigma_1=\left(1-\frac{2}{n}\right)(\rho-\sigma)$$

$$\rho_2-\sigma_2=\left(1-\frac{2}{n}\right)(\rho_1-\sigma)$$

$$=\left(1-\frac{2}{n}\right)^2\cdot(\rho-\sigma)$$

....

....

Also $$\rho_m-\sigma_m=\left(1-\frac{2}{n}\right)(\rho_{m-1}-\sigma_{m-1})$$

$$=\left(1-\frac{2}{n}\right)^m(\rho-\sigma)\text{.........III}$$

and $$\rho_m+\sigma_m=\rho_{m-1}+\sigma_{m-1}=\ldots\ldots=\rho+\sigma\text{......IV}$$

Solving III and IV, we get

$$\rho_m=\rho+\frac{\sigma-\rho}{2}\left[1-\left(1-\frac{2}{n}\right)^m\right]$$

And $$\sigma_m=\sigma+\frac{\rho-\sigma}{2}\left[1-\left(1-\frac{2}{n}\right)^m\right]$$

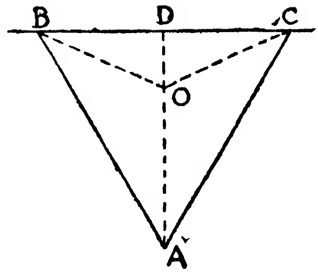
19. If h_1 be the height of cylinder under consideration, and x is depth to which this cylinder is immersed.

$$\therefore \quad x=\sigma h_1$$

Water will be on the point of over-flowing when the volume of this depth x be just equal to one quarter of the outside cylinder. Therefore,

$$\pi r_1^2\,.\,x=\frac{1}{4}\,\pi r^2 h$$

$$\therefore \quad h_1=\frac{x}{\sigma}=\frac{r^2 h}{4\sigma\,.\,r_1^2}.$$



20. Let the required point is O. Since the depths of the angular points of the $\triangle$OBA and $\triangle$OCA are the same. Hence if the pressures are to be the same, their areas must be equal.

$$\therefore \quad \triangle BOA = \triangle OAC$$

From this we conclude that O lies on AD (the median). Therefore

$$\frac{1}{3} = \frac{\text{Pressure on } \triangle \text{ BOC}}{\text{Pressure on } \triangle \text{ BAC}}$$

$$= \frac{\text{Area of } \triangle \text{ BOC} \times \text{depth of its C. G.}}{\text{Area of } \triangle \text{ BAC} \times \text{depth of its C. G.}}$$

$$= \frac{OD \times OD}{DA \times DA}$$

$$= \frac{OD^2}{DA^2}$$

$$\therefore \quad \frac{OD}{DA} = \frac{1}{\sqrt{3}}$$

21. Suppose ρ and σ be the densities of the cone and liquid. It is given that $\frac{3}{4}$ of the axis of the cone is immersed. Therefore,

$$\sigma \times \left(\frac{3}{4}\right)^2 = \rho \,.\, 1$$

$$\therefore \quad \frac{\sigma}{\rho} = \frac{64}{27}$$

Let the radius of the base of the cone is a, and which is the radius of the cylinder. Therefore, volume of the water which is above the vertex of the cone

$$= \pi a^2 \,.\, \frac{3h}{4} - \frac{1}{3}\pi \,.\, \frac{3h}{4}\left(\frac{3a}{4}\right)^2.$$

If the surface fall through a distance y when the cone is removed, therefore this must be equal

$$=\pi a^2\left(\frac{3h}{4}-y\right)$$

Equating both

$$\pi a^2 \cdot \frac{3h}{4}-\frac{1}{3}\pi \cdot \frac{3h}{4}\left(\frac{9a^2}{64}\right)=\pi a^2 \cdot \frac{3h}{4}-\pi a^2 y$$

$$y=\frac{9h}{64}.$$

22. Thrust on the plane surface of the hemisphere

$$=\pi \cdot 3^2 \cdot 6w$$
$$=54\,\pi w.$$

Resultant vertical thrust on the hemisphere

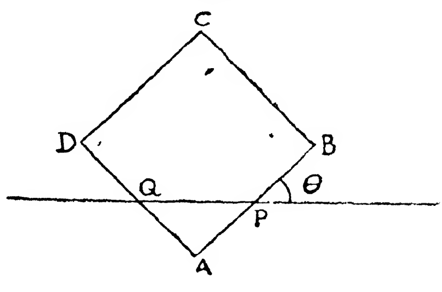
$$=\text{weight of displaced liquid}$$
$$=\frac{2}{3}\pi\,(3)^3 \cdot w=18\pi w$$

Required thrust $=\sqrt{(54\,\pi w)^2+(18\,\pi w)^2}$

$$=18\,\pi w\,\sqrt{9+1}$$
$$=18\pi w\sqrt{10}$$

making an angle $\tan^{-1}\frac{1}{3}$ to the horizontal.

23. Let ABCD be the square, A being the lowest point. Let $AB=a$; $AP=x$ and $AQ=x\tan\theta$



Let ρ=density of rods

σ=density of liquid.

Therefore,

$$(x+x\tan\theta)\,\sigma=4a\cdot\rho\ldots\ldots\ldots\text{I}$$

Taking moment about A, we get

$$\left(x\cdot\frac{x}{2}\cos\theta-x\tan\theta\cdot\frac{x\tan\theta}{2}\cdot\sin\theta\right)\sigma$$

$$= 4a\rho \cdot \tfrac{1}{2}\, a\sqrt{2} \cdot \cos(45+\theta)$$
$$= 2a^2\rho\,(\cos\theta - \sin\theta)$$
$$\therefore \quad x^2\,(1-\tan^3\theta)\,\sigma = 4\,a^2\rho\,((1-\tan\theta) \ldots\ldots\ldots \text{II}$$

From I and II, we get

$$\tan^2\theta\,(\sigma-4\rho)+2\tan\theta\,(\sigma-2\rho)+\sigma-4\rho=0 \ldots\ldots \text{III}$$

This Eqn. will have real roots, when

$$(\sigma-2\rho)^2-(\sigma-4\rho)^2 \text{ is positive}$$
$$(4\rho\sigma-12\rho^2) \text{ is positive}$$
$$\text{i. e. } \sigma > 3\rho.$$

If $\sigma > 4\rho$, from III, the value of $\tan\theta$ found would be negative, which is clearly impossible. If $\sigma = 4\rho$, then Eqn, III will give

$$\tan\theta = 0\ ; \quad x = a$$

which is the limiting case when a diagonal is horizontal.

$$\therefore \quad \sigma > 3\rho \text{ and } < 4\rho$$

24. From the condition of problem

$$W' + \frac{W}{s} = \text{tension of the chain} = W - \frac{W}{s}$$

$$s = \frac{2W}{W-W'}$$

weight of the water not $> W'$

$$\therefore \quad \text{Volume of the water not} > \frac{W'}{W}$$

$$\therefore \quad \frac{\text{Volume of the water}}{\text{Volume of the weight } W} \not> \frac{\frac{W'}{W}}{\frac{W}{sW}} \not> \frac{sW'}{W'}$$

$$\text{and therefore} \quad \not> \frac{2W'}{W-W'}$$

25. Let σ_1, σ_2 be the sp. gr. of the water and cylinder, ρ and ρ' those of air.

Let x and x' be the portions of the cylinder immersed in the two cases, h is the height. Therefore,

$$h\sigma_2 = x\sigma_1 + (h-x)\,\rho$$
$$h\sigma_2 = x'\sigma_1 + (h-x')\,\rho'$$
$$\therefore \quad x - x' = h\left(\frac{\sigma_2-\rho}{\sigma_1-\rho}\right) - h\left(\frac{\sigma_2-\rho'}{\sigma_1-\rho'}\right)$$
$$= h\,\frac{(\sigma_1-\sigma_2)\,(\rho'-\rho)}{(\sigma_1-\rho)\,(\sigma_1-\rho')} = +\text{ve}$$

since $\rho' > \rho$.

If $x > x'$, the cylinder will rise in water

If the surface of water inside is lowered than that of outside by z, where $\Pi' - \Pi = wz$

$$\therefore \quad z = \frac{\Pi}{w}\left(\frac{\Pi'}{\Pi} - 1\right)$$
$$= \text{H}\,.\left(\frac{\rho'-\rho}{\rho}\right)$$

where H is the height of the water barometer originally. Hence the cylinder goes down through z and up through $(x-x')$

It will fall if $z > x - x'$

$$\frac{\text{H}}{\rho} > h\,\frac{\sigma_1-\sigma_2}{(\sigma_1-\rho)\,(\sigma_1-\rho')}$$

Hence in general, it will fall.

26. Let AB is inclined at an angle θ to the horizon and if we take moments about A for the two rods, we see that $\tan\theta = 1$; therefore, portion of equilibrium is a symmetrical one.

Let ρ and σ be the densities of the rod and water

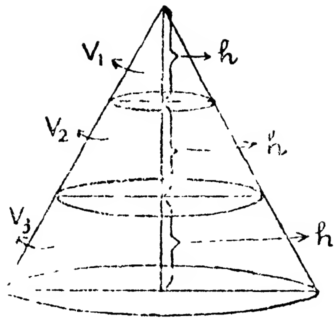
Resolving vertically

$$a\rho = b\sigma \ldots\ldots\ldots \text{I}$$

Taking moment about A for one rod,

$$2a\,\rho w\,\frac{a}{\sqrt{2}} = 2b\sigma w\,.\,\frac{b}{\sqrt{2}} + \text{T}\,.\,\frac{2a}{\sqrt{2}} \ldots\ldots\ldots \text{II}$$
$$\therefore \quad \text{T} = (a-b)\,\rho w$$
$$= \frac{a-b}{2a}\,.\,w$$

27. Let V_1, V_2, V_3 be the volumes of the portions as shown in figure



$\therefore \quad V_1 : V_2 : V_3$

$= h^3 : \{(2h)^3 - h^3\} : \{(3h)^3 - (2h)^2\}$

i. e. as $1 : 7 : 19$

Let ρ be the density of the cone and $\sigma_1 - \sigma_2$, σ_1, and $\sigma_1 + \sigma_2$ those of liquids.

$$\therefore \quad 27\rho = 19\,(\sigma_1 - \sigma_2) + 7 \,.\, \sigma_1 + (\sigma_1 + \sigma_2)$$

$$27\rho = 7\,(\sigma_1 - \sigma_2) + 19\sigma_1$$

$$27\rho = 27\sigma_1 - 18\sigma_2 = 26\sigma_1 - 7\sigma_2$$

$$\therefore \quad \sigma_1 = 11\sigma_2 \quad \text{and} \quad \rho = \frac{31}{3}\,\sigma_2$$

$$\therefore \quad \sigma_1 - \sigma_2 = 10\sigma_2$$

$$\sigma_1 + \sigma_2 = 12\sigma_2$$

Hence the reqd. ratio

$$31 : 30 : 33 : 36$$

28. If the depth be x,

Stretched length of the string $= a + x$

its tension $= nw \,.\, \dfrac{x}{a}$

where w is the weight of the cylinder.

Therefore, for Equilibrium

$w =$ tension of the string $+$ weight of the fluid displaced

$$= n\,w \,.\, \frac{x}{a} + \frac{x}{h} \,.\, \frac{w}{\sigma}$$

$$\therefore \quad x = \frac{h\,a\sigma}{nh\sigma + a}$$

29. Let the height of the water barometer be h. Pressure inside the cylinder $= (h + 1700)$ cm.

From Boyle's law, we have

$$1200000 \times h = 450000 \times (h + 1700)$$

$$\therefore \quad h=1020$$

$$\therefore \quad P=w\times(1700+1020)$$
$$=2720\times 980 \text{ dynes.}$$

30. Let the free surface, a parabola of latus-rectum $\frac{2g}{w^2}$, meet the fixed leg in A and the revolving leg in P. Draw PN perp. to the fixed leg. Therefore,

$$c^2=\frac{2g}{w^2}\,.\,\text{AN} \quad \text{where} \quad \text{AN}=\frac{w^2c^2}{2g}.$$

Let the heights of A and P above the bottoms of the legs be h, k.

$$\text{Mean level}=\frac{\sigma_1 h+\sigma_2 k}{\sigma_1+\sigma_2}$$

$$\text{Reqd. result}=k-\frac{\sigma_1 h+\sigma_2 k}{\sigma_1+\sigma_2}$$
$$=\frac{\sigma_1(k-h)}{\sigma_1+\sigma_2}$$
$$=\frac{\sigma_1}{\sigma_1+\sigma_2}\,.\,\frac{w^2c^2}{2g}$$

31. If the sp. gr. of the plank be σ

$$W+W=\frac{2}{3}\,.\,\frac{W}{\sigma}\,\ldots\ldots\ldots\ldots\ldots\text{I}$$

Let the man walk to the end A of the plank, the section through the length of the plank being ABCD, B and C being under the water and D above. Draw AE horizontal to cut CD in E. Therefore $\triangle$AED must be $\frac{1}{3}$ of ABCD. Therefore

$$\text{DE}=\frac{2b}{3} \quad \therefore \quad \tan\theta=\frac{2b}{3a}$$
$$\text{where } \theta=\angle\text{DAE}$$

The weight of the plank is W acting at the centre B of the section.

Taking moment about A

$$W\left(\frac{1}{\sigma}-1\right)\left(\frac{a}{2}\cos\theta+\frac{b}{2}\sin\theta\right)$$

$$=\text{moment of }\triangle\text{AED about A}$$

$=$sum of moments of weights each $\frac{1}{3}\cdot\frac{W}{\sigma}$ at each pts. A, E, D

$$=\frac{1}{3}\cdot\frac{W}{\sigma}\cdot\frac{a\cos\theta+(a\cos\theta+\text{DE}\sin\theta)}{3}$$

Hence

$$\left(\frac{1}{\sigma}-1\right)\left(\frac{a}{2}+\frac{b}{2}\tan\theta\right)=\frac{1}{9\sigma}\cdot\left(2a+\frac{2b}{3}\tan\theta\right)$$

$$\therefore\quad \frac{9a^2}{2}+3b^2=\frac{1}{\sigma}\left[\frac{5a^2}{2}+\frac{23b^2}{9}\right]$$

From I

$$\frac{w}{W}=\frac{2}{3}\cdot\frac{1}{\sigma}-1=\frac{6\,(9a^2+6b^2)}{45a^2+46b^2}-1$$

$$=\frac{9a^2-10b^2}{45a^2+46b^2}$$

Hence the Limiting value is that

$$\frac{w}{W}\text{ is not }>\frac{9a^2-10b^2}{45a^2+46b^2}.$$

32. Let the radius of the base of the cone is r and h is the height. Let b be the radius of the section of the cone by the water line and c the radius of the section of the cylinder

$$\therefore\quad b^2=\frac{6}{19}c^2\ldots\ldots\ldots\ldots\text{I}$$

When the cone is in equilibrium, let the level of water rises by x inches. Therefore,

$$\pi c^2\,.\,x-\frac{1}{3}\,\frac{\pi a^2h}{h^3}\left[\left(\frac{b}{a}h+x\right)^3-\left(\frac{bh}{a}\right)^3\right]$$

$$=\frac{19}{24}\pi\,\frac{b^3h}{a}$$

i. e. $$\frac{19b^2x}{6}-\frac{a^2}{3h^2}\left[\left(\frac{bh}{a}+x\right)^3-\left(\frac{bh}{a}\right)^3\right]=\frac{19}{24}\cdot\frac{b^3h}{a}$$

Put $x=\frac{bh}{a}\cdot y$

We have $$(1+y)^3-1=\frac{19}{2}(y-\tfrac{1}{4})$$

$$\therefore\quad y=\frac{1}{2}$$

$$\therefore\quad x=\frac{bh}{2a}$$

The additional buoyancy of the water is

$$=\text{weight of }\frac{1}{3}\cdot\frac{\pi a^2h}{h^3}\left[\left(\frac{bh}{a}+x\right)^3-\left(\frac{bh}{a}\right)^3\right]$$

$$=\text{weight of }\frac{19}{24}\cdot\frac{\pi b^3h}{a}\text{ of water}$$

which is balanced by the additional weight of water poured into the cone.

Hence in equilibrium.

33. Inside air has a pressure due to 51 feet. Let x cubic feet air be pumped into the bell. With the usual formula. we have

$$\frac{51\times84}{1+\alpha\,.\,7}=\frac{34\times(x+84)}{1+\alpha\,.\,0}$$

$$\therefore\quad x+84=\frac{273}{280}\cdot\frac{51\times84}{34}=122{\cdot}85$$

$$x=122{\cdot}85-84$$
$$=38{\cdot}85.$$

34. With the usual notation,

$$T=W-w\,.\,Ax\ldots\ldots\ldots\ldots\ldots\text{I}$$

Also $$x^2+(a+h)x-hb=0\ldots\ldots\ldots\ldots\text{II}$$

Eliminating x between I and II

$$(W-T)^2+wA\,(a+h)\,(W-T)-w^2A^2\,hb=0$$

Put $W=n\cdot wA$, $T=w\cdot AY$, and $X=a$, we get

$$Y^2-2XY-(2n+h)Y+nX+n^2+nh-hb=0$$

This curve is a hyperbola.

35. Let the bell be sunk at a distance x.

From Boyle's law

$$\frac{2}{3}\pi a^3(x-C+H)=(\frac{2}{3}\pi a^3+\pi a^2c)\times H$$

$$\therefore \quad x=c\left[1+\frac{3H}{2a}\right]\ldots\ldots\ldots\ldots\text{I}$$

Let V be the volume. Therefore,

$$\left(V+\frac{2}{3}\pi a^3+\pi a^2c\right)\times H=\left(\frac{2}{3}\pi a^3+\pi a^2c\right)\left(H+x\right)$$

$$\therefore \quad V=\left(\frac{C}{H}+\frac{36}{2a}\right)\left(\frac{2}{3}\pi a^3+\pi a^2c\right)$$

which gives the reqd. result.

36. Let BC be the horizontal and CB the vertical leg. Let P, Q be the ends of the Mercury in BC, CB, when there is equilibrium, so that $OP=CQ=d$. The pressure at Q being II. Let A be the vertex corresponding to the free surface which goes through P. Therefore

$$II=\tfrac{1}{2}w^2\rho\cdot l^2-\rho g\cdot AN$$

$$O=\tfrac{1}{2}w^2\rho d^2-\rho g\cdot AO$$

where QN is perp. to AO.

$$II=\tfrac{1}{2}w^2\rho(l^2-d^2)-\rho g\cdot ON$$

$$\rho g(h+d)=\tfrac{1}{2}w^2\rho(l^2-d^2)$$

$$\therefore \quad w^2=\frac{2g(h+d)}{l^2-d^2}$$

i. e. $$d^2+\frac{2g}{w^2}d+\frac{2gh}{w^2}-l^2=0$$

This Eqn. will give positive root only if

$$\frac{2gh}{w^2}-l^2 \text{ is negative}$$

i. e. $w^2 > \frac{2gh}{l^2}$.

37. Let the tube revolve about tangent at O, the end of the horizontal radius. PCR being the vertical radius, P the highest point. Let the free surface cut the tube in Q, having the vertex at A, a point below O. Therefore

$$\angle QCR = \theta$$

Draw PM, QN perp. to tangent at O.

Therefore, $a^2 = PM^2 = \frac{2g}{w^2} \cdot AM$

Also $(a - a \sin\theta)^2 = QN^2 = \frac{2g}{w^2} \cdot AN$

Hence, subtracting,

$$a^2 - a^2 (1 - \sin\theta)^2 = \frac{2g}{w^2} \cdot MN$$

$$\therefore \quad a^2 [2 \sin\theta - \sin^2\theta] = \frac{2ga}{w^2} (1 + \cos\theta)$$

$$= \frac{2ga}{w^2} \cdot 2\cos^2\frac{\theta}{2}$$

i. e. $aw^2 \left(\tan\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) = g.$

38. Let C be the centre of the vessel and A is the lowest point. Draw a parabola, latus rectum $\frac{2g}{w^2}$, with vertex A and axis vertical. Let these meet the circle in point P and P′. Let PP′ meet AC producde in N. Let $\angle$ PCN$=\theta$. Therefore

$$r^2 \sin^2\theta = \frac{2g}{w^2} \cdot AN = \frac{2gr}{w^2} (1 + \cos\theta)$$

$$\therefore \quad 1 - \cos\theta = \frac{2g}{w^2 r} \quad \ldots\ldots\ldots\ldots\ldots \text{I}$$

But $n^3 \cdot \frac{4}{3}\pi a^3$=volume of the liquid.

=volume of spherical portion PAP'

--volume of paraboloided portion PAP'

$$=\frac{\pi}{3}(a+a\cos\theta)[3a^2-(a^2-a^2\cos\theta+a^2\cos^2\theta)]$$

$$-\tfrac{1}{2}\pi a^2\sin^2\theta\,.\,(a+a\cos\theta)$$

Therefore

$$\tfrac{4}{3}n^3=\tfrac{1}{3}(1+\cos\theta)(2+\cos\theta-\cos^2\theta)-\tfrac{1}{2}\sin^2\theta\,(1+\cos\theta)$$

$$=\tfrac{1}{6}(1+\cos\theta)^3$$

$$\therefore\quad 2n=1+\cos\theta=2-\frac{2g}{w^2r}$$

$$w^2=\frac{g}{r(1-n)}$$

If $w^2>\frac{g}{r(1-n)}$; the liquids break into two i. e. a hole. Hence there would not allow any liquid to escape.

39. Let V is the lowest point of the cone, N the highest point of the axis, and A is the vertex of the free surface. Therefore,

$$V^2=\frac{2g}{w^2}\,.\,AN=\rho\,.\,AN$$

Take any point Q on the surface of the conical section. Draw QM perp. to the axis and let $QM=y$. Therefore

$$\frac{P}{\rho}=\frac{w^2}{2}\,.\,QM^2+g\,.\,AM$$

$$=g\left(\frac{y^2}{l}+AM\right)$$

$$\therefore\qquad AM=h+r\cot\alpha-VM-AN$$

where h=height of the cylinder

$$\therefore\qquad AM=h+n\cot\alpha-y\cot\alpha-\frac{r^2}{l}$$

$$\therefore \quad \frac{\rho}{P} \cdot \frac{l}{g} = y^2 + l \cot \alpha (r - y) + lh - r^2$$

$$= \left(y - \frac{l}{2} \cot \alpha\right)^2 + lr \cot \alpha + lh - r^2 - \frac{l^2 \cot^2 \alpha}{4}$$

This is least when

$$y = \frac{l}{2} \cot \alpha$$

This value of y will give a point on the cone if

$$\frac{l}{2} \cot \alpha < r$$

$$\text{i. e.} \quad l < 2r \tan \alpha$$

Hence the proportion.

40. Let the vertex of the cone is V and the axis of the cone cut the base in COD, O being the centre. Let the free surface meet the axis VB in A, VC, VD in P and P′. Let PP′ cut the axis in N. Let VN=y. Therefore

$$y^2 \tan^2 \alpha = \frac{2g}{w^2} \cdot \text{AN} = \frac{3h}{4} \tan^2 \alpha \cdot \text{AN}$$

$$\therefore \quad \text{AN} = \frac{4y^2}{3h}$$

Therefore $\quad \frac{1}{4} \times \frac{1}{3} \pi h^3 \tan^2 \alpha = \text{volume VPAP}'$

$$= \tfrac{1}{3} \pi y^3 \tan^2 \alpha + \pi y^2 \tan^2 \alpha \cdot \frac{4y^2}{2 \times 3h}$$

$$\therefore \quad 8y^4 + 4hy^3 - h^4 = 0$$

$$\therefore \quad y = h/2$$

Therefore, $\quad \text{OA} = h - \frac{h}{2} - \text{AN} = \frac{h}{2} - \frac{h}{3} = \frac{h}{6}$.

Let the vertical line cut the parabola in Q and Q′ and let QQ′ meet the axis in M. Therefore,

$$h^2 \tan^2 \alpha = \frac{2g}{w^2} \cdot \text{AM} = \frac{3h}{4} \tan^2 \alpha \cdot \text{AM}$$

$$\therefore \quad AM=\frac{4h}{3}$$

$$\therefore \quad OM=\frac{4h}{3}+\frac{h}{6}=\frac{3h}{2}$$

Thrust on the base

$$=\pi h^2 \tan^2 \alpha \,.\, OM \,.\, w-\pi h^2 \tan^2 \alpha \,.\, \tfrac{1}{2}AM \,.\, w$$

$$=\pi h^2 \tan^2 \alpha \,.\, w\left(\frac{3h}{2}-\frac{2h}{3}\right)$$

$$=\tfrac{5}{6}\pi h^3 \tan^2 \alpha \,.\, w$$

$$\frac{\text{Thrust on base}}{\text{weight of water}}=\frac{\frac{5}{6}}{\frac{3}{4}\times\frac{1}{3}}=\frac{10}{3}.$$

41. Let the free surface has the vertex at A and passing through P and Q the ends of the liquid. Since the tube is half, PQ will pass through C (centre). Draw PM, QN perp. to the axis. Therefore

$$QN^2=\frac{2g}{w^2}\,.\,AN$$

$$PM^2=\frac{2g}{w^2}\,.\,AM$$

$$\therefore \quad QN^2-PM^2=\frac{2g}{w^2}\,.\,MN$$

$$2P\,(QN-PM)=\frac{2g}{w^2}\,.\,MN$$

$$\therefore \quad \tan\theta=\frac{QN-PM}{MN}=\frac{g}{Pw^2}$$

$$\therefore \quad \theta=\tan^{-1}\frac{g}{Pw^2}.$$

42. If we revolve semi-circle BDC about its axis OD, hemisphere will be formed. Through B and C draw the free parabolic surface, having vertex at A,

$$\therefore \quad a^2=OC^2=\frac{2g}{w^2}\,.\,OA.$$

Paraboloid BAC=half hemisphere

$$\therefore \quad \frac{1}{2} \text{ OA} \times \pi a^2 = \frac{1}{3} \pi a^3$$

$$\therefore \quad \frac{1}{2} \cdot \frac{w^2 a^2}{2g} = \frac{1}{3} a$$

$$\therefore \quad w = \sqrt{\frac{4g}{3a}}.$$

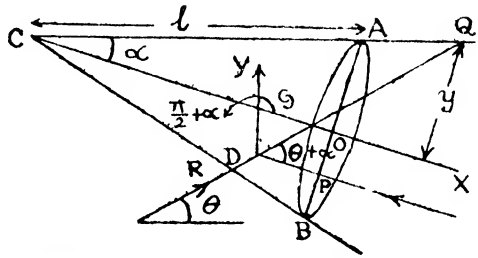
43. Height of the cone

$$h = l \cos \alpha$$

radius of the base, $r = l \sin \alpha$

$$\therefore \quad \text{OG} = \frac{l \cos \alpha}{4}$$

and $\text{OP} = -\dfrac{l \sin \alpha}{4}$



Take O, the centre of the base, as origin, OC as the x-axis and AB as the y-axis.

Then the equation of the line CA is

$$\text{Y} - \text{O} = \tan \alpha \, (x - l \cos \alpha)$$

or $\quad x \sin \alpha - y \cos \alpha = l \sin \alpha \, . \cos \alpha \ldots\ldots\ldots\ldots \text{I}$

Now the gradient (m) of the line of action R is $\tan(\theta + \alpha)$

$$= \frac{1}{\cot(\theta + \alpha)} = \frac{1}{2 \tan \alpha} = \frac{\cos \alpha}{2 \sin \alpha}$$

Also it passes through D where ordinate is $\text{OP} = -\dfrac{l \sin \alpha}{4}$

whose abscissa can be found by putting $y = -\dfrac{l \sin \alpha}{4}$ in the Eqn. to GD

But the Equation to GD is

$$\text{Y} - \text{O} = \tan\left(\frac{\pi}{2} + \alpha\right)\left(x + \frac{l \cos \alpha}{4}\right)$$

$$x \cos \alpha + y \sin \alpha = -\frac{l \cos^2 \alpha}{4}$$

Hence the abscissa of D is given by

$$x\cos\alpha-\frac{l\sin\alpha}{4}\,.\sin\alpha=-\frac{l\cos^2\alpha}{4}$$

$$\text{i. e.}\quad x\text{D}=-\frac{l}{4\cos\alpha}$$

$\therefore$ The Equation to the line of action of R is

$$y+\frac{l\sin\alpha}{4}=\frac{\cos\alpha}{2\sin\alpha}\left(x+\frac{l\cos 2\alpha}{4\cos\alpha}\right)$$

$$\text{or}\quad \frac{4y+l\sin\alpha}{4}=\frac{4x\cos\alpha+l\cos 2\alpha}{8\sin\alpha}$$

$$\text{or}\quad 8y\sin\alpha+2l\sin^2\alpha=4x\cos\alpha+l\cos 2\alpha$$

$$4(x\cos\alpha-2y\sin\alpha)=\frac{l}{4}(2\sin^2\alpha-\cos 2\alpha)\ldots\ldots\text{II}$$

Eliminating x between Eqn. I and II, we have

$$y(-\cos^2\alpha+2\sin^2\alpha)=l\left[-\sin\alpha\,.\cos^2\alpha-\frac{\sin^3\alpha}{2}+\frac{\sin\alpha\,.\cos 2\alpha}{4}\right]$$

Hence $\text{CQ}=y\operatorname{cosec}\alpha$

$$=\frac{3l}{4(1-3\sin^2\alpha)}$$

Magnitude of thrust is

$$\text{R}=\pi r^3 w\cos\alpha\sqrt{1+\tfrac{1}{9}\operatorname{cosec}^2\alpha-\tfrac{2}{3}}$$

$$=\tfrac{1}{3}\pi r^3 w\cos\alpha\sqrt{3+\operatorname{cosec}^2\alpha}$$

$$=\tfrac{1}{3}\pi r^3\cot\alpha\sqrt{1+3\sin^2\alpha}\,.w.$$

But $r=l\sin\alpha$

$$\therefore\quad \text{R}=\tfrac{1}{3}\pi l^3\sin^2\alpha\,.\cos\alpha\,.\sqrt{1+3\sin^2\alpha}\,.w\,.$$

44. Let A be the fixed point, C the centre of the plane base, B the point at which the hemisphere is loaded so that $\angle\text{ACB}=90°$. Let the vertical through G the centre of gravity meet the plane base in N. We have a downward force nw at B.

$$\text{Upward force}=\text{W}\left(\frac{1}{\sigma}-1\right)\text{ at G.}$$

$$\angle\,\text{CGN}=\theta$$

Taking moment about A,

$$AB \cdot n = AN\left(\frac{1}{\sigma} - 1\right)$$

$$\therefore \quad AN = \frac{\sigma}{1-\sigma} nAB$$

$$= \lambda na\sqrt{2}$$

$$\therefore \quad CN^2 = CA^2 + AN^2 - 2CA \cdot AN \cos 45$$

$$= a^2 (1 + 2\lambda n + 2\lambda^2 n^2)$$

$$\therefore \quad \tan\theta = \frac{CN}{CG} = \frac{CN}{\frac{3}{8} a}$$

$$= \frac{\frac{8}{3}\sqrt{1 - 2\lambda n + 2\lambda^2 n^2}}{\frac{3}{8} a}$$

$$\text{If} \quad \sigma = \frac{1}{1+n}$$

$$\lambda = \frac{\sigma}{1-\sigma} = \frac{1}{n}$$

$$\tan\theta = \frac{8}{3}$$

$$\therefore \quad \cos\theta = \frac{3}{\sqrt{73}}$$

45. Let the plane through the axis VO cut the cone in VC and VD. Let the diameter of the base is AB for Equilibrium.

Upward pressure on the cone $\not> W$, weight of the cone i. e.

$$\pi a^2 h \cdot w - \tfrac{1}{3}\pi a^2 hw \not> w$$

$$\tfrac{1}{3}\pi a^2 hw \not> \tfrac{1}{2} w$$

Reqd. ratio $\not> :\frac{1}{2}$

Take K in VO

$$\therefore \quad VK = \tfrac{3}{4} VO$$

Resultant horizontal thrust through it

$$= ahw \cdot \frac{2h}{3}$$

Moment about V of this thrust

$$=\frac{2ah^2\,w}{3}\times\frac{3h}{4}=\frac{1}{2}ah^3w$$

Draw vertical lines CC′, BB′, DD′

Moment about V of the vertical thrust on the half cone

$$=\frac{\pi a^2}{2}\,wh\times OG'-\frac{1}{6}\,\pi a^2\,wh\,.\,\frac{3}{4}\,.\,OG'$$

$$=\tfrac{3}{8}\,.\,\pi a^2\,w\,.\,h\,.\,OG'$$

where G′ is on OB and is the C. G. of the semi-circle CGD and therefore

$$OG'=\frac{4a}{3\pi}$$

Therefore moment of the vertical thrust

$$=\frac{a^3\,wh}{2}$$

Dist. of C. G. of the arc $BD=\frac{2a}{\pi}$ from O ;

Moment of the weight of half-shell about V

$$=\frac{w}{2}\times\frac{2}{3}\,.\,\frac{2a}{\pi}=\frac{2}{3}\,.\,\frac{wa}{\pi}$$

For Equilibrium

$$\frac{1}{2}\,ah^3w+\frac{a^3\,wh}{2}\not>\frac{2}{3}\,\frac{wa}{\pi}$$

$$\frac{1}{3}\,\pi a^2\,hw\left(\frac{h^2}{a^2}+1\right)\not>\frac{4}{3}\,w$$

i. e. weight of water $(1+\cot^2\alpha)\not>\frac{4}{9}\,w$

$$\frac{\text{weight of water}}{w}\not>\frac{4}{9}\sin^2\alpha$$

46 Weight nw is placed at B on the point of the rim. Since it is the maximum weight, the water line must pass through B. Let α be the inclination of OB to the horizon where O is the centre.

Taking moment about O,

$$OB\ \cos\alpha\ .\ nw = OG\ \sin\alpha\ .\ w$$
$$= \tfrac{1}{2}OB\ \sin\alpha\ .\ w$$
$$\therefore\quad \tan\alpha = 2n.$$

The volume of the portion of a sphere of radius a which is cut off by a plane at a distance x from the centre is

$$= \frac{\pi}{3}\ (a-x)^2\ .\ (2a+x)\quad \text{where } x = a\sin\alpha.$$

Therefore, resolving vertically

$$(n+1)\ w = \frac{\pi}{3}\ (a - a\sin\alpha)^2\ .\ (2a + a\sin\alpha)$$
$$= \frac{2\pi}{3}\ a^3\ .\ w \times \tfrac{1}{2}\ (1-\sin\alpha)^2\ .\ (a+\sin\alpha)$$
$$\frac{w}{\frac{2\pi}{3}\ a^3\ w} = \frac{(1-\sin\alpha)^2\ (2+\sin\alpha)}{2\ (n+1)}$$

(which is the reqd. result)

47. With the usual Eqn.

$$x^2 + (a+h)\ x - hb = 0$$

Solving for x, we have

$$x = \frac{-(a+h) + \sqrt{(a+h)^2 + 4hb}}{2}$$

The pressure inside is $= (x+a+h)$ height of water height corresponding to Mercury barometer

$$= \frac{x+a+h}{\sigma} = \frac{a+h+\sqrt{(a+h)^2+4hb}}{2\sigma}$$

Let a block of wood of volume V and sp. gr. ρ be floated and the length of the air column inside the bell be y. Let V′ be the volume of the portion of the wood not in water

Therefore, we have

$$(Ay - V')\ (y+a+h) = Ah\ .\ b$$

i. e. $\left(y - \dfrac{V'}{A}\right)(y+a+h) = h\ .\ b$ I

Also $x\ (x+a+h) = h\ .\ b$ II

Hence the height of the barometer is increased, since it measures a water pressure $(y+a+h)$ instead of $(x+a+h)$

If the block fall from a shelf within the bell and floats, so that V' of it is out of the water and z is the length now corresponding to x. Therefore

$$(Az-V')\ (z+a+h)=(Ax-V)\ (x+a+h)$$

It follows that z cannot be $> x$; for if it were, then

$$z+a+h > x+a+h$$
$$Az-V' > Ax-V$$
$$V' < V$$
$$\therefore \quad z < x$$

Hence the barometer falls.

48. If d be the depth of the dock, H is the height of the water-barometer, b the height of the bell. Therefore,

$$(d+H-b+h)\ .\ h=b\ .\ H,$$

neglecting small quantities

$$h\ (d+H)=b\ .\ H \qquad \ldots\ldots\ldots \text{I}$$

Let x be the length occupied by the air when the lowest point of the bell is at a depth y, therefore

$$x\ (y+H)=bH=h\ (d+H) \qquad \ldots\ldots\ldots \text{II}$$

Let the weight of the bell be W, and V its volume,

$$W=Aaw\sigma=V\rho w \qquad \ldots\ldots\ldots \text{III}$$

Tension of the chain

$$=Aaw\sigma-[Ax+V]\ w\sigma$$
$$=Aw\sigma\left[a\left(1-\frac{\sigma}{\rho}\right)-x\right]$$
$$=Aw\sigma\left[a\left(1-\frac{\sigma}{\rho}\right)-\frac{h\ (d+H)}{y+H}\right]$$

When this tension is zero, the bell will rise

$$(y+H)\ a\left(1-\frac{\sigma}{\rho}\right)=h\ (d+H)$$

y will be positive when

$$h\ (d+\mathrm{H}) > \mathrm{H}a\left(1-\frac{\sigma}{\rho}\right)$$

$$\frac{d}{\mathrm{H}} > \frac{a}{h}\left(1-\frac{\sigma}{\rho}\right)-1$$

49. Let the length air occupied be x, when the bell is owered. Pressure inside it$=(x+a+h)$ due to a height of water. Specific gravity of the air

$$=\frac{x+a+h}{h}\sigma$$

Let the volume of the body be V

$$(\mathrm{V}-\mathrm{CA})\ .\ 1+\mathrm{C}\ .\ \mathrm{A}\ .\ \sigma=\text{weight}$$

$$=[\mathrm{V}-(c+r\sigma)\ \mathrm{A}]\ .\ 1+\mathrm{A}\ (c+r\sigma)\ \frac{x+a+h}{h}\sigma$$

i. e. $(x+a)\ (c+r\sigma)=hr\ (1-\sigma)$ I

Also $(b-c)\ h=(x-c-r\sigma)\ (x+a+h)$ II

Eliminate x from I and II

Treat σ as negligible, from I we have

$$(x+a)\ c=rh$$

Substituting in II

$$(b-c)\ h=\left(\frac{rh}{c}-a-c\right)\times h\left(\frac{r}{c}+1\right)$$

$$\therefore\quad (r+c)\ (rh-ac-c^2)=bc^2-c^3$$

i. e. $r^2h+cr\ (h-a-c)-c^2\ (a+b)=0$

50. If BP be the tube, let B be the lowest point, Q any point on the tube. Draw QN, QM perp. to the vertical hrough B. Let BQ$=x$, BP$=l$.

Pressure at P

$$\Pi=\mathrm{P}\ [\tfrac{1}{2}w^2\ .\ \mathrm{PN}^2-g\ .\ \mathrm{AN}]$$

$$\mathrm{P}=\text{pressure at Q}$$

$$=\rho\ [\tfrac{1}{2}w^2\ \mathrm{QM}^2-g\ .\ \mathrm{AM}]$$

$$\therefore\quad \frac{\mathrm{P}-\Pi}{\rho}=\tfrac{1}{2}w^2\ \sin^2\alpha\ (x^2-l^2)-g\ (x-l)\cos\alpha$$

$$=\frac{w^2 \sin^2 \alpha}{2}\left[x-\frac{g \cos \alpha}{w^2 \sin^2 \alpha}\right]^2-\frac{w^2 l^2}{2} \cdot \sin^2 \alpha+gl \cos \alpha-\frac{g^2 \cos^2 \alpha}{2w^2 \sin^2 \alpha}$$

P is least when

$$x=\frac{g \cos \alpha}{w^2 \sin^2 \alpha}$$

This least value of P should not be negative, otherwise there will be a vacuum in the tube. Therefore,

$$\frac{\Pi}{\rho}-\frac{w^2 l^2 \sin^2 \alpha}{2}+gl \cos \alpha-\frac{g^2 \cos^2 \alpha}{w^2 \sin^2 \alpha} \text{ must be } +\text{ve}.$$

Hence $$\frac{w^2 l^2 \sin^2 \alpha}{2}-gl \cos \alpha<\frac{\Pi}{\rho}-\frac{g^2 \cos^2 \alpha}{2w^2 \sin^2 \alpha}$$

$$\therefore \quad \frac{w^2 \sin^2 \alpha}{2}\left[l-\frac{g \cos \alpha}{w^2 \sin^2 \alpha}\right]^2<\frac{\Pi}{\rho}$$

$$l<\frac{g \cos \alpha}{w^2 \sin^2 \alpha}+\frac{\sqrt{2\Pi \rho}}{\rho w \sin \alpha}$$

$$<\frac{g \rho \cos \alpha+w \sin \alpha \sqrt{2\Pi \rho}}{w^2 \rho \cdot \sin^2 \alpha}$$

51. Let the vertical section through the axis cut the free surface of the liquid in the parabola QPAP′Q′, Q and Q′ being the points in which it meets the vessel, A the vertex. Draw QMQ′, PWP′ perp. to the common axis.

$$R^2=\frac{2g}{w^2} \cdot AM$$

$$r^2=\frac{2g}{w^2} \cdot AN$$

$$AM=\frac{w^2 R^2}{2g}$$

$$AN=\frac{w^2 r^2}{2g} \qquad \ldots\ldots\ldots \text{I}$$

Let the height of the solid cylinder be h and t is the height of A above its lowest position.

From the condition of equilibrium

$$\pi r^2 h\rho = \pi r^2 \left(t + \tfrac{1}{2}\,\mathrm{AN}\right)$$

$$\therefore \qquad h\rho = t + \frac{w^2 r^2}{4g} \quad \ldots\ldots\ldots\ldots \mathrm{II}$$

Let y be the height of the lowest point of the solid cylinder from the bottom of the vessel and $\pi R^2 b$ be the total volume of water

$$\pi R^2 b = \pi R^2 y + \pi\,(R^2 - r^2)\,(t + \mathrm{AN})$$
$$+ [\pi R^2 \,.\, \mathrm{NM} - \tfrac{1}{2}\,\pi R^2 \,.\, \mathrm{AM} + \tfrac{1}{2}\,\pi r^2 \,.\, \mathrm{AN}] \ldots\ldots \mathrm{III}$$

$$\therefore \quad R^2 \,.\, b = R^2 y + h\rho\,(R^2 - r^2) + \frac{w^2}{4g} \,.\, R^2\,(R^2 - r^2) \ldots\ldots \mathrm{IV}$$

If y_0 be the value of y initiallly when $w=0$

$$R^2 \,.\, b = R^2 \,.\, y_0 + h\rho\,(R^2 - r^2) \ldots\ldots\ldots \mathrm{V}$$

From IV and V we have

$$y_0 - y = \frac{w^2}{4g}\,(R^2 - r^2)$$

53. We know that $P = g\rho h$

$$= \frac{32 \times 13 \times 1000}{16} \times \frac{5}{2} \text{ poundals}$$

Let x be the required measure, we have

$$x\,[M']\,[L']\,[T']^{-2} = \frac{32 \times 13 \times 1000 \times 5}{2 \times 16}\,[M]\,[L]^{-1}\,[T]^{-2}$$

$$\therefore \quad x = 13 \times 1000 \times 5 \times \frac{[M]}{[M']} \times \frac{[L']}{[L]} \times \frac{[T']^2}{[T]^2}$$

$$= 13 \times 1000 \times 5 \times \frac{1}{\frac{1}{16}} \times 3 \times (60)^2$$

$$= 11232 \times 10^6.$$

54, We can divide the depth z into a large number of equal portions (say n).

Value of gravity at a depth $\frac{rz}{n}$ is $a+b\,.\,\frac{rz}{n}$. Pressure due an element at a depth $\frac{z}{n}$ is

$$=\rho\left(a+b\,.\,\frac{rz}{n}\right)\times\frac{z}{n}$$

Hence the pressure at a depth z

$$=\Sigma\,\rho z\left(\frac{a}{n}+\frac{bz}{n^2}\,.\,r\right)$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\,\rho z\left[\frac{na}{n}+\frac{bz}{n^2}\,.\,\frac{n(n+1)}{2}\right]$$

$$=\rho\,[az+\tfrac{1}{2}\,bz^2]$$

55. Let the cylinder is floating at a depth x_0 immersed. Therefore

$$x_0\,.\,\rho a=l\sigma\,.\,a$$

$$\therefore\quad x_0=\frac{l\sigma}{\rho}$$

If P be the force that would be required to immerse a depth x, then

$$x=x_0+\frac{r}{n}\,(1-x_0)$$

Also $\quad x\rho\,ag=\mathrm{P}+l\sigma\,ag$

$$\therefore\quad \mathrm{P}=ag\,(x\rho-l\sigma)$$

$$=a\rho g\,(x-x_0)$$

$$=a\rho g\,.\,\frac{r}{n}\,(l-x_0)$$

The work done P

$$=\mathrm{P}\times\frac{1}{n}\,(l-x_0)$$

$$=a\rho g\,\frac{r}{n^2}\,(l-x_0)^2$$

Total work

$$=\sum_{n=a} a\rho g\,(l-x_0)^2\,\frac{1+2+\ldots\ldots+n}{n^2}$$

$$=\Sigma a\rho g\quad(l-x_0)^2\times\frac{\frac{1}{2}n\,(n+1)}{n^2}$$

$$=\tfrac{1}{2}a\rho g\,(l-x_0)^2$$

$$=\tfrac{1}{2}gal^2\,\frac{(\rho-\sigma)^2}{\rho}$$

57. Let the rod float vertically with $2b$ of its length in a liquid of density $n\rho$, and a weight w attached to its lower end.

$$2bgn\rho=2ag\rho+w$$

$$\text{i. e.}\quad nb=a+\frac{w}{2g\rho}\qquad\ldots\ldots\ldots\text{I}$$

$2a$ is the length of the rod, and ρ the density of the rod.

If the rod is inclined at a small angle to the vertical, upward thrust$=2bgn\rho$ acting at a dist. b from the lower end, downward thrust$=2ag\rho$ acting at a dist a from the lower end.

For stable Equilibrium,

$$2bg\,.\,n\,.\,\rho\,.\,\frac{b}{2}>2ag\rho\,.\,a$$

$$\therefore\quad nb^2>a^2$$

$$nb>a\sqrt{n}$$

Therefore from I,

$$a+\frac{w}{2g\rho}>a\sqrt{n}$$

$$\therefore\quad w>(\sqrt{n}-1)\,2g\rho a$$

$$w>(\sqrt{n}-1)\ \text{(its weight)}$$

(which is the result).

58. Let the axis turned through an angle θ, which is very small, and h' the length of the axis then immersed. Weight of the liquid displaced

$$=\frac{h'^3}{h^3}\cdot 2\mathrm{W}$$

where the length of the axis is h.

$$\therefore \quad \frac{h'^3}{h^3}\cdot 2\mathrm{W}=\mathrm{W}+w$$

$$\therefore \quad \frac{h'}{h}=\left(\frac{\mathrm{W}+w}{2\mathrm{W}}\right)^{1/3}=(\tfrac{1}{2})^{1/3} \text{ approx.}$$

Height of the metacentre above the vertex of the cone

$$=\tfrac{3}{4}h'+\frac{3a^2}{4h^2}\cdot h'$$

$$=\tfrac{3}{2}h' \quad \text{where } a=h.$$

Taking moment about the vertex

$$(\mathrm{W}+w)\cdot\frac{3h'}{2}\sin\theta=\tfrac{3}{4}h\sin\theta\ .\ \mathrm{W}+w\ (a\cos\theta+h\sin\theta)$$

Neglecting second order product of $w\theta$

$$\mathrm{W}\ .\ 2h'\theta=h\ .\ \theta\ .\ \mathrm{W}+\tfrac{4}{3}wa$$

$$\therefore \theta\left[\frac{2h}{2^{\frac{1}{3}}}-h\right]=\frac{4wa}{3\mathrm{W}}$$

$$\theta\ [(4)^{1/3}-1]=\frac{4w}{3\mathrm{W}}$$

$$\therefore \quad \theta=\frac{4w}{3\ (4^{1/3}-1)\ \mathrm{W}}$$

59. In equilibrium let the length x of the cylinder be immersed in the outside fluid

$$x\rho=h'\sigma+nh\rho \qquad \text{.........I}$$

If M, M′ be the metacentres for the outside and inside liquids. If O be the lowest point of the axis

$$\therefore \quad OM=\frac{x}{2}+\frac{a^2}{4x}$$

$$OM'=\frac{h'}{2}+\frac{a^2}{4w}$$

Thrust upwards at $M=\pi a^2 x\rho$

Thrust downwards at $M'=\pi a^2 h'\sigma$

Thrust downwards at $G=\pi a^2 \,.\, nh\rho$

For stability

$$x\rho\left(\frac{x}{2}+\frac{a^2}{4x}\right) > h'\sigma\left(\frac{h'}{2}+\frac{a^2}{4h'}\right)+nh\rho \,.\, \frac{h}{2}$$

i. e. $2x^2\rho > (2h'^2+a^2)\sigma+(2nh^2-a^2)\,\rho$

From I

$$2\,[h'\sigma+nh\rho]^2 > (2h'^2+a^2)\,\rho\sigma+(2nh^2-a^2)\,\rho^2.$$

60. For Equilibrium, if the length x of the axis of the cone be immersed in the outer liquid. Therefore,

$$\frac{1}{3}\pi x^3 \tan^2\alpha \,.\, \rho=\frac{1}{3}\pi h'^3 \tan^2\alpha \,.\, \sigma+\frac{1}{3}\pi\rho n\,.^3 \tan^2\alpha$$

$$\therefore \quad x^3\rho=h'^3\sigma+h^3n\rho\ldots\ldots\ldots \text{I}$$

Let M, M′ be the metacentres for the cutside and inside liquids. Let the vertex of the cone be V.

$$\therefore \quad VM=VH+HM=\frac{3}{4}x+\frac{3}{4}x\tan^2\alpha$$

$$=\frac{3}{4}x\sec^2\alpha$$

$$VM'=\frac{3}{4}h'\sec^2\alpha$$

Thrust upwards at $M=\frac{1}{3}\pi\tan^2\alpha \,.\, x^3\rho$

Thrust downwards at $M'=\frac{1}{3}\pi\tan^2\alpha \,.\, h'^3\sigma$

Thrust downwards at $G = \frac{1}{3} \pi \tan^2\alpha \, . \, nh^3\rho$

For stability

$$x^3 \rho \times \frac{3}{4} x \sec^2\alpha > h'^3\sigma \times \frac{3}{4} h' \sec^2\alpha + nh^3 \rho \times \frac{2h}{3}$$

$$\therefore \quad x^4\rho > h'^4\sigma + \frac{8}{9} nh^4 \rho \cos^2\alpha$$

From I, we get

$$(h'^3\sigma + h^3 n\rho)^4 > \rho \left[h'^4\sigma + \frac{8}{9} nh^4 \rho \cos^2\alpha \right]^3$$